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Estimation and Control of Hybrid Systems

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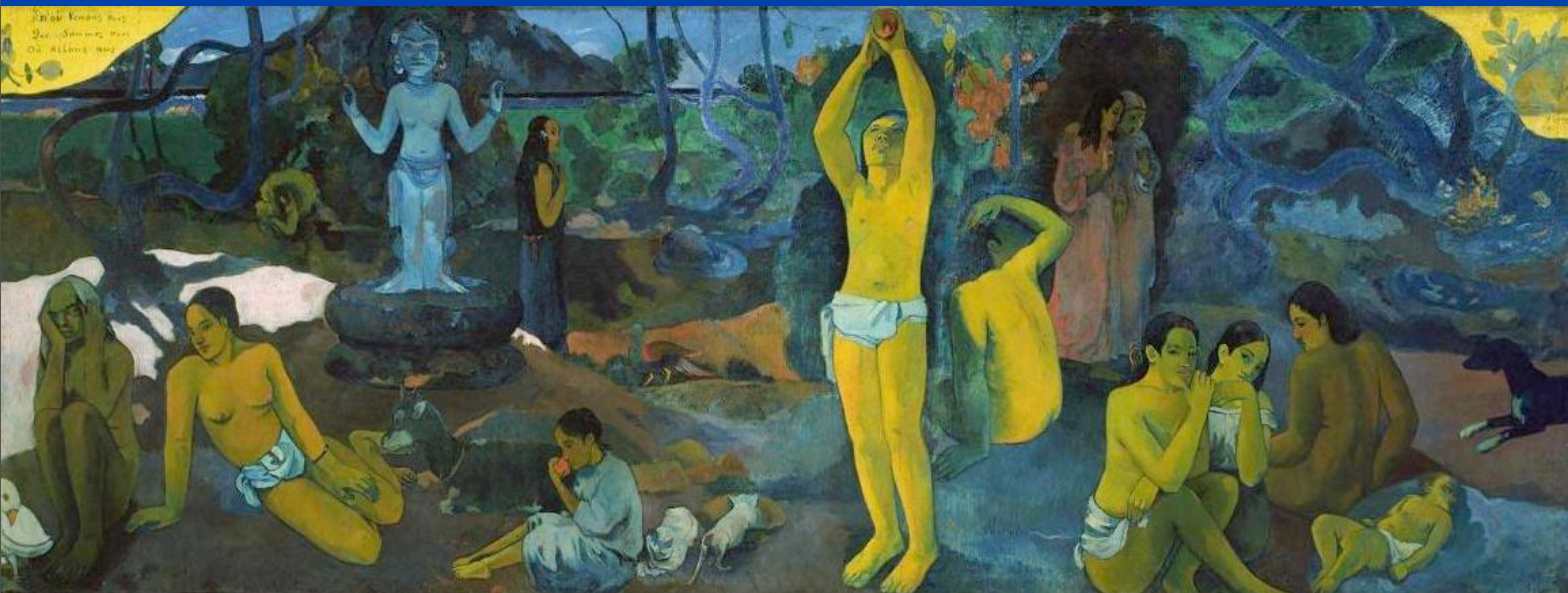
27/05/2008



"D'où Venons Nous / Que Sommes Nous / Où Allons Nous"

Paul Gaugin (Museum of Fine Arts, Boston, Massachusetts, USA)

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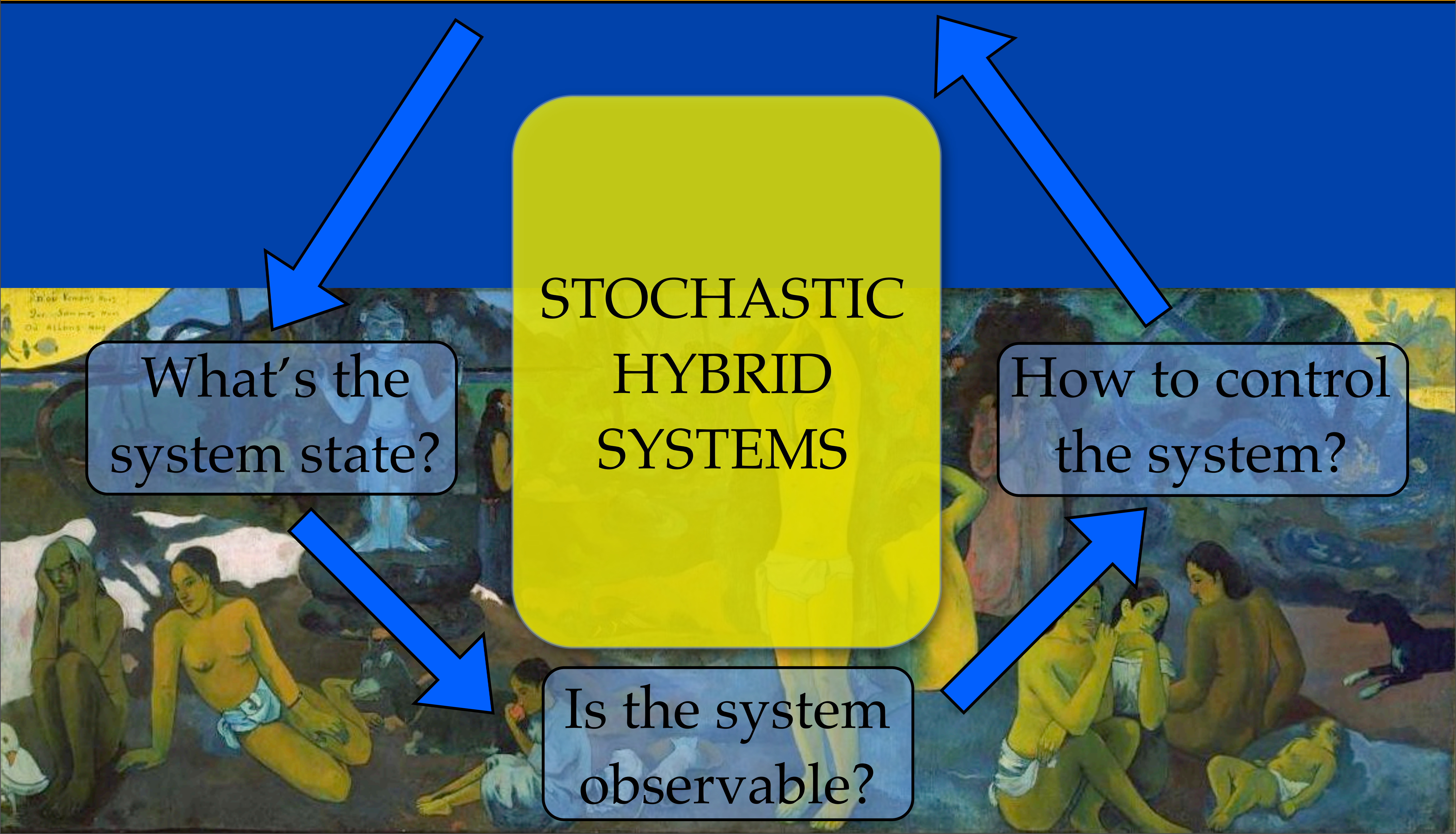
Context...

ESTIMATION & CONTROL



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Motivation...

Why study hybrid systems?

World is full of complex interconnected systems

↓

Sophisticated software and hardware onboard

↓

Embedded systems

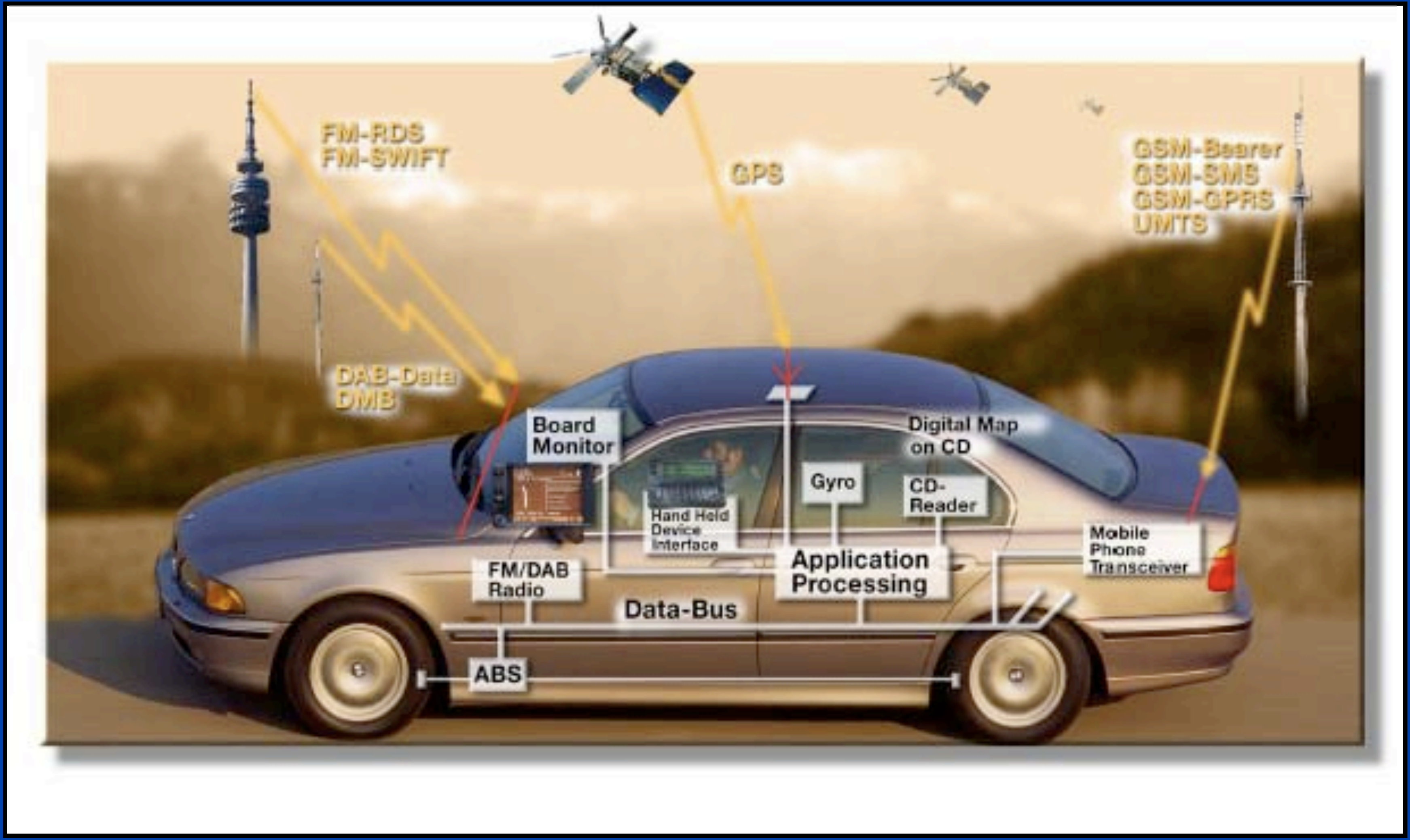
↓

Continuous signals + discrete events

Motivation...

Automotive industry

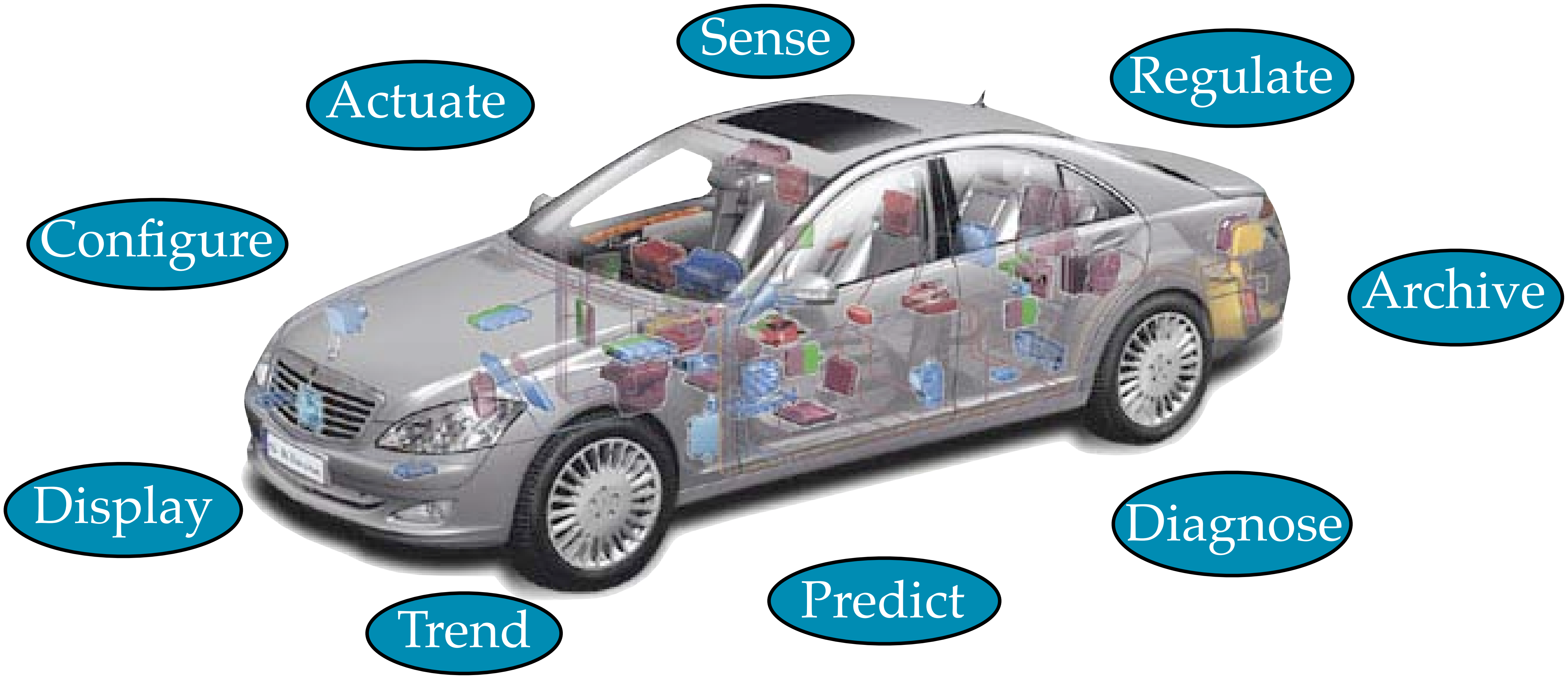
- Cost of a car comes more than 30% from Electronics.
- More than 80 microprocessors and millions of lines of code.
- 90% of future innovations will be based on electronic systems.



Motivation...

Automotive industry

- Product specification (communications protocols).
- System integration and critical software development.



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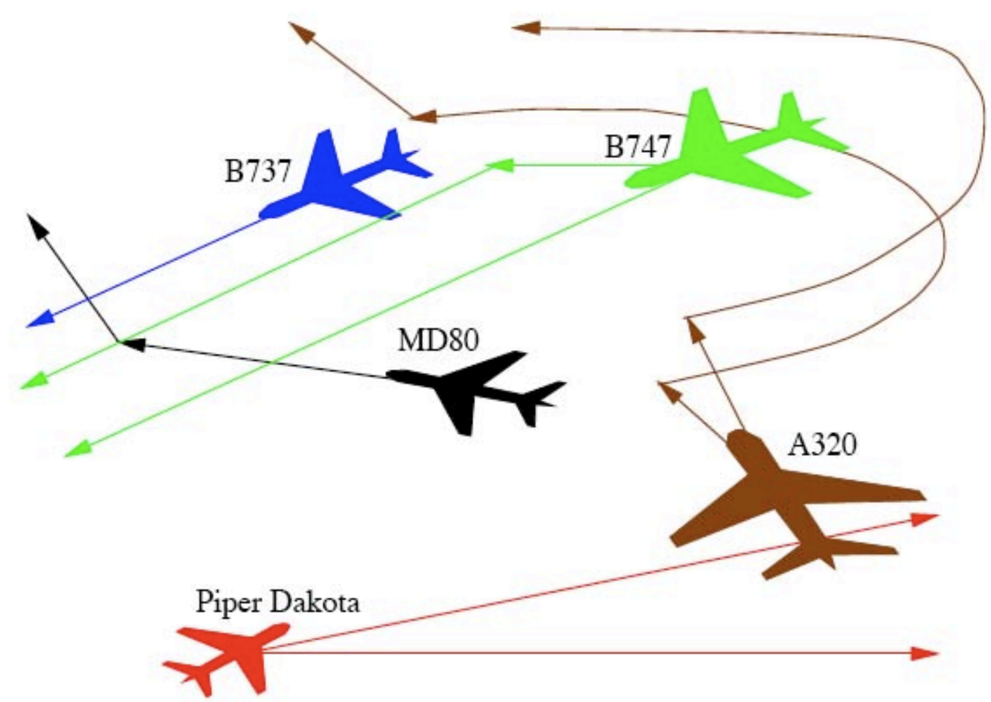
Motivation...

Traffic management

- Automated highway systems: *platoon control.*
- Air traffic management.
- Power management.
- Large scale multi-agent systems: *cooperative control.*



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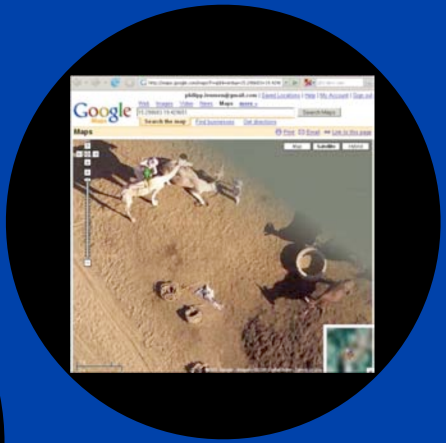
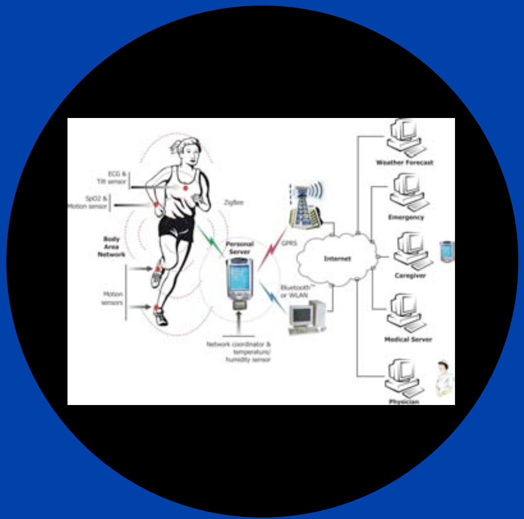
Motivation...

Communications

- The world is becoming wireless!
- From main stream servers to personal devices (mobile phone, pda,...).
- Shared and adaptive communications networks.
- Heterogeneous hardware/software, mixed architectures.
- New applications (toy industry, e-commerce, voip,...).





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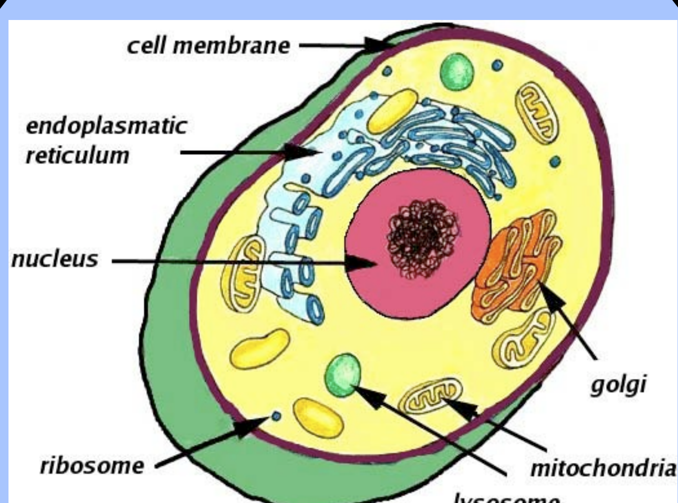
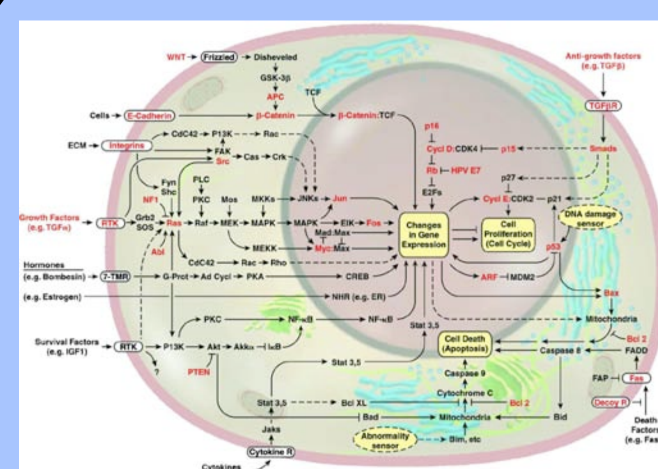
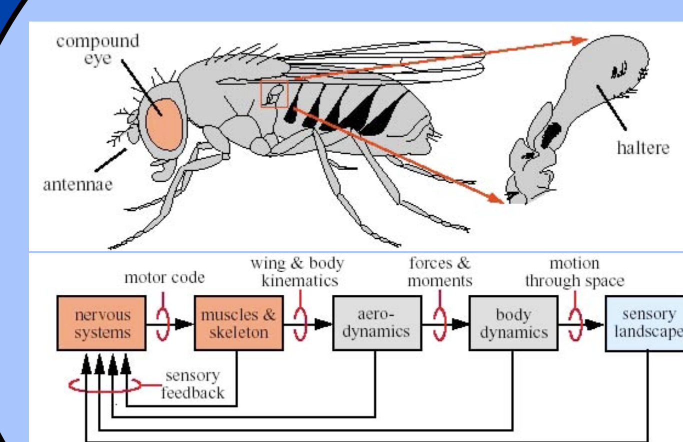
Motivation...

Systems Biology

- Understand the knowledge system level of a biological system.
- Principles from control help understand biological systems.
- Biological systems provide a rich source of examples for control.
- Medical advances, new drugs, gene therapies, biomedical research.







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Motivation...

Why study hybrid systems?

- Modeling **abstraction** of a wide range of systems:

  - Systems with phased operation (walking robots, systems with colisions)

  - Systems controlled by discrete inputs (switches, valves, digital computers)

  - Hierarchical coordinating systems (multi-agent)

- Merge of **computation + physics + communications**, the core of new technological innovations:

  - Automated Highway Systems

  - Air Traffic Management Systems

  - Safety systems

  - Biological systems

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Motivation...

Why study estimation and control of hybrid systems?

1) State estimation enables fault detection!

Detect critical situations → Avoid error propagation

2) Control algorithms require full state feedback!

Measuring is not economically feasible or physically possible → Observers are needed

Rather complex and still partially unsolved problem

Outline...

I. Modeling of hybrid systems

II. Estimation of stochastic hybrid systems

III. Optimal control of stochastic hybrid systems

IV. Experimental application

V. Conclusions and future developments

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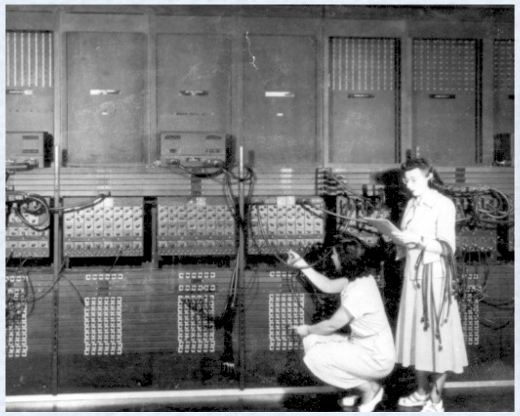
Part I

MODELING OF HYBRID SYSTEMS

Modeling of hybrid systems



Computer Science

Control Theory

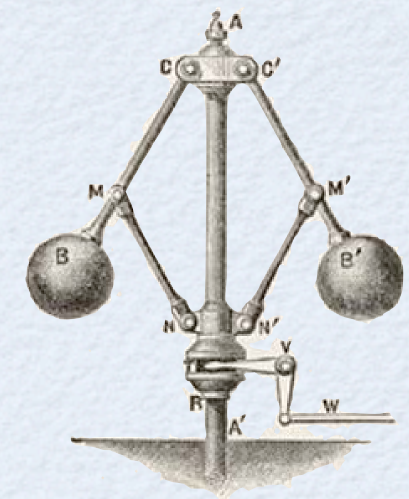
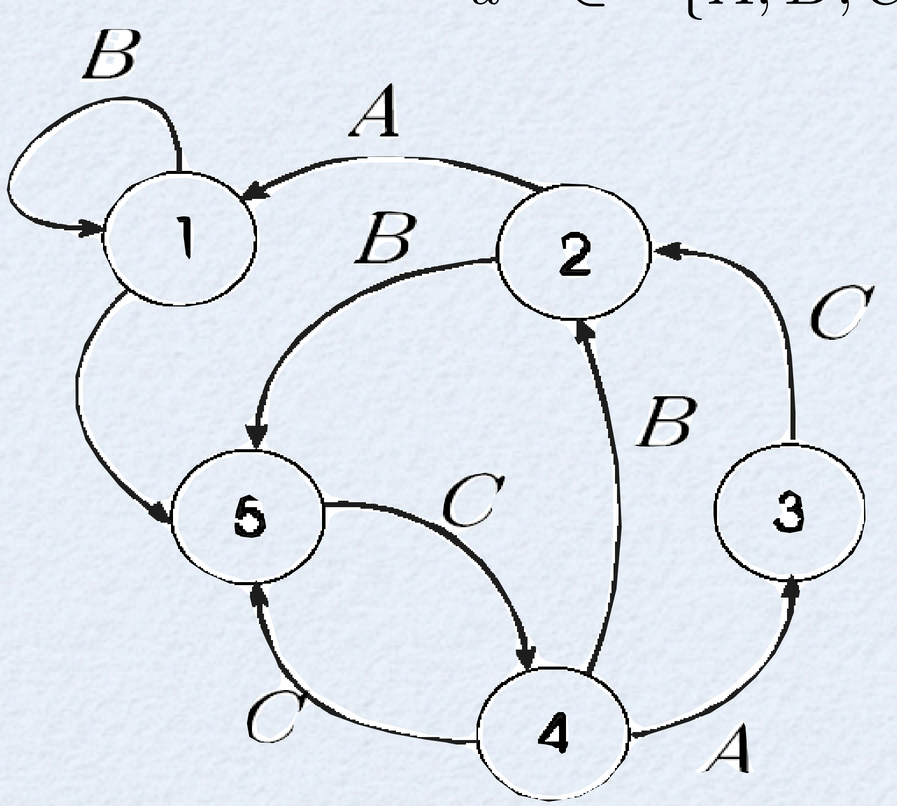


Fig. 29.—The Governor.

$$x \in \{1,2,3,4,5\}$$
$$u \in \{A,B,C\}$$

$$x \in \mathbb{R}^n$$
$$u \in \mathbb{R}^m$$
$$y \in \mathbb{R}^p$$



HYBRID SYSTEMS

$u(t)$ →

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = g(x(t), u(t))$$

→ $y(t)$

Finite state machines

Continuous dynamical systems

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Modeling of hybrid systems

Event-driven world

computer science

Tends to abstract from the physical world

Time-driven world

control theory

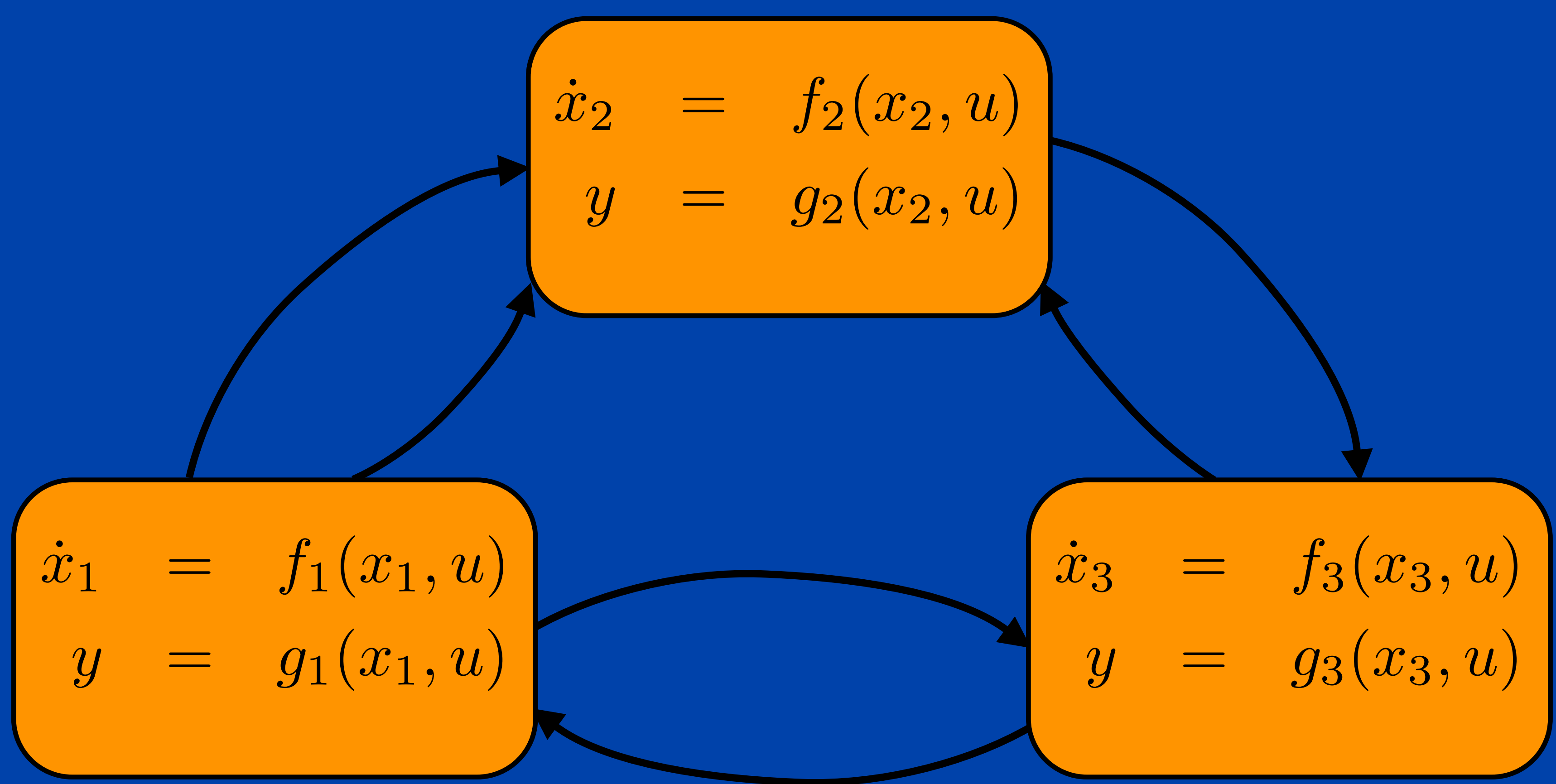
Tends to ignore computational limitations

Objective 1: **descriptive enough** to capture the system behaviour

Objective 2: **simple enough** for analysis and synthesis problems

Modeling of hybrid systems

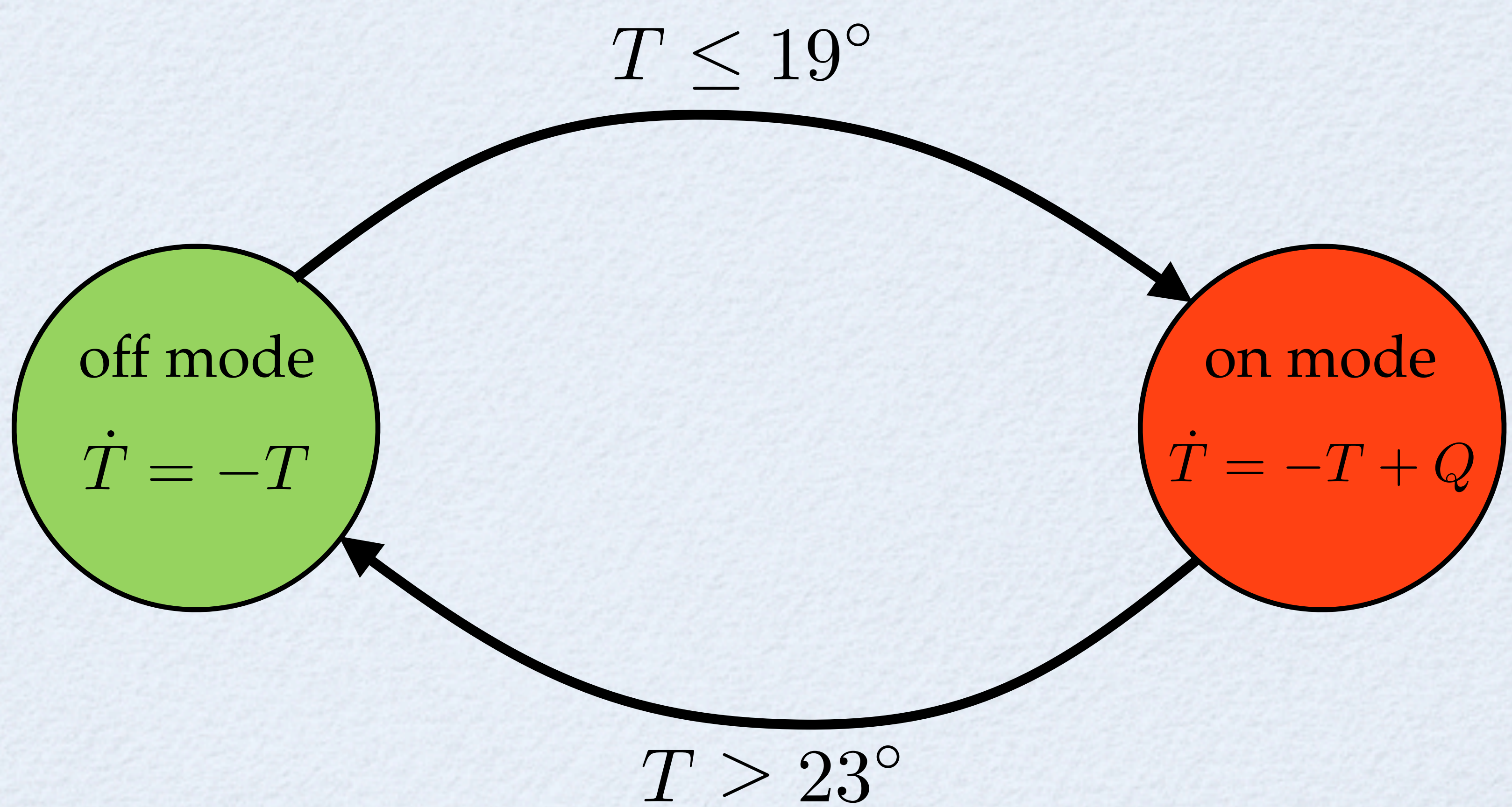
- System can be in one of several modes (Discrete Mode).
- Each mode behavior described by difference/differential equations.
- Switching between modes due to occurrence of events:
  - *external/internal signals, or system dynamics itself.*



Modeling of hybrid systems

Intrinsically hybrid systems...

Hybrid dynamics:



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Modeling of hybrid systems

Intrinsically hybrid systems...

4 stroke engine

Discrete input + Continuous input + Continuous states

- Discrete input → valves
- Continuous input → fuel, air
- Continuous states → pressure, temperature, ...

Modeling of hybrid systems

Intrinsically hybrid systems...

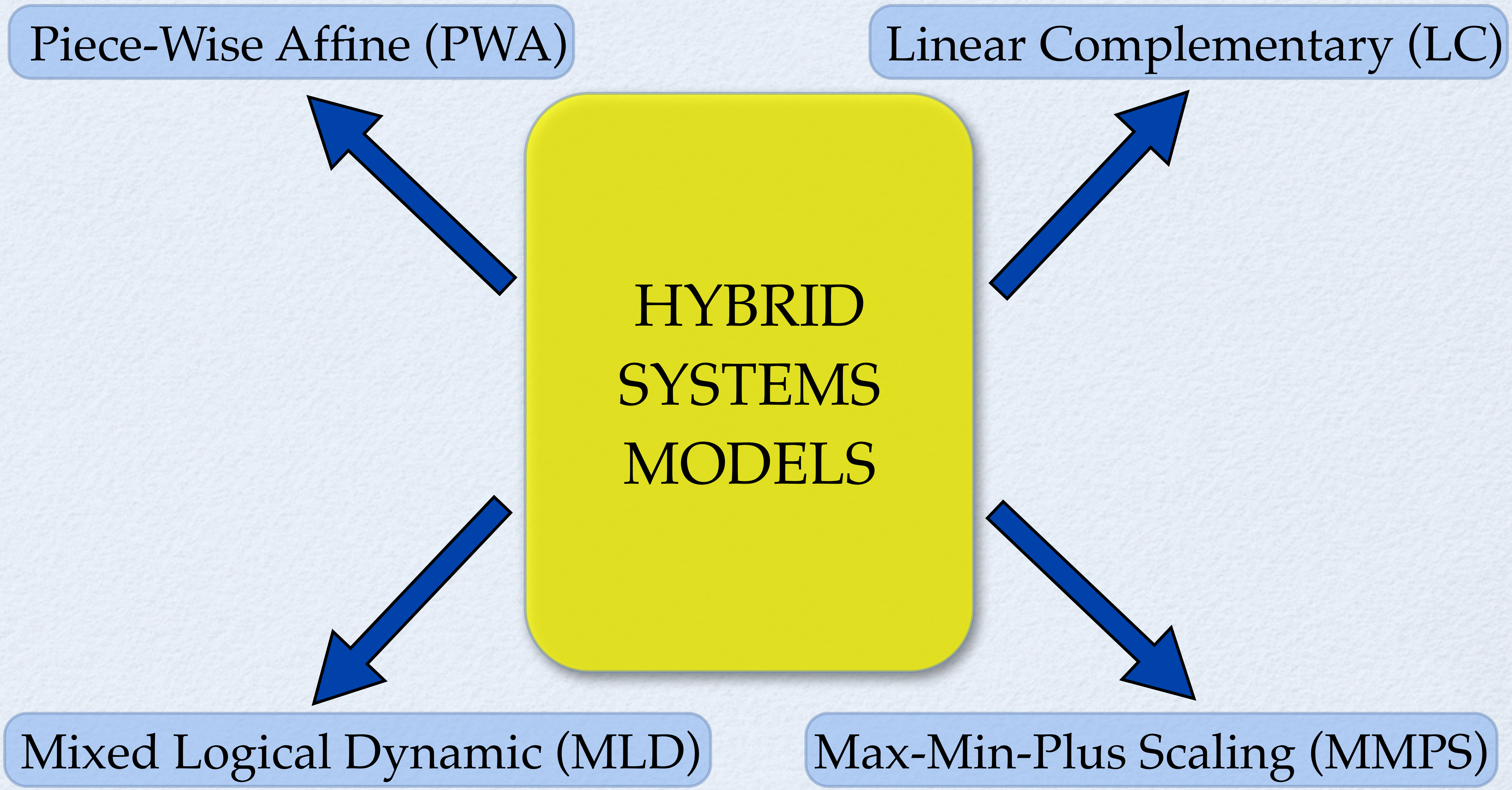
Driving a motorcycle

Discrete input + Continuous input + Continuous states

- Discrete input → 1,2,3,4,N
- Continuous input → brakes, gas, clutch
- Continuous states → velocity, torque, ...

Modeling of hybrid systems

Representation of hybrid systems...



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Modeling of hybrid systems

The Piece-Wise Affine (PWA) model

$$x(k+1) = A_{i(k)}\, x(k) + B_{i(k)}\, u(k) + f_{i(k)}$$
$$i(k) = j \quad \text{iff} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \Omega_j$$
$$y(k) = C_{i(k)}\, x(k) + D_{i(k)}\, u(k) + g_{i(k)}$$

$$i(k) \in \mathcal{I} \triangleq \{1, \ldots, s\} \subset \mathbb{N}^+ \,, \forall_k$$

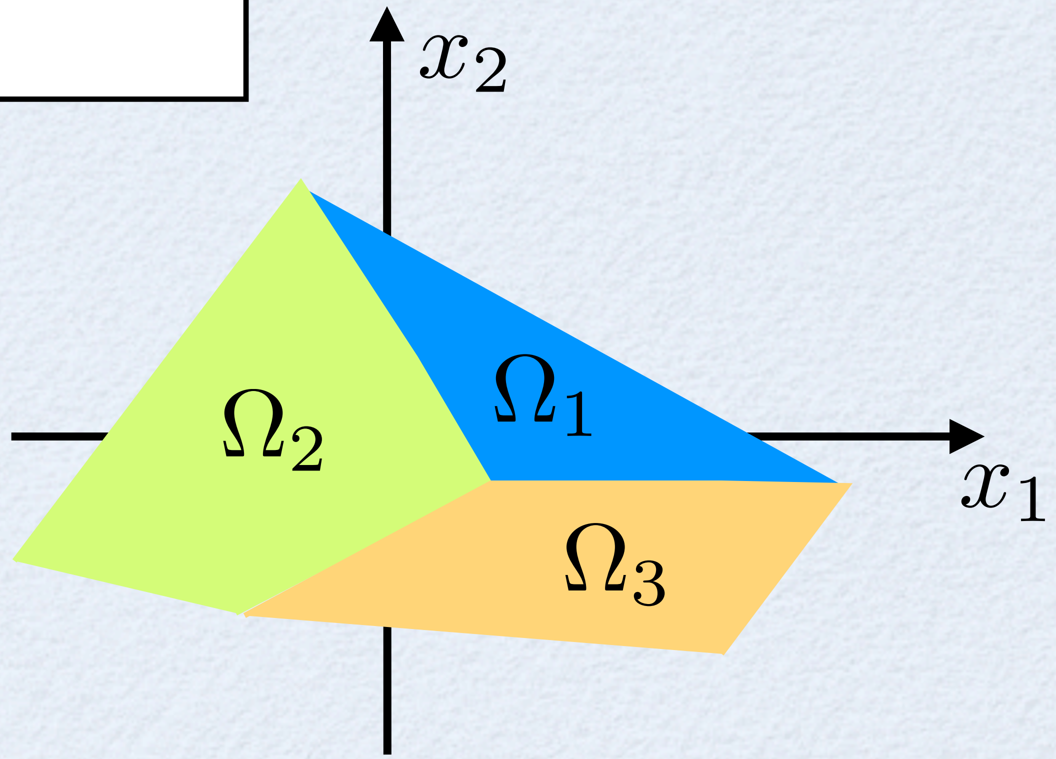
Non-overlapping regions:

$$i \neq j \;\Rightarrow\; \Omega_i \bigcap \Omega_j = \emptyset$$
$$\boldsymbol{\Omega} \triangleq \bigcup_{i \in \mathcal{I}} \Omega_i$$

Polytopes definition in the input+state region:

$$S_i\, x(k) + R_i\, u(k) \leq T_i$$

Ω is also a polytope.

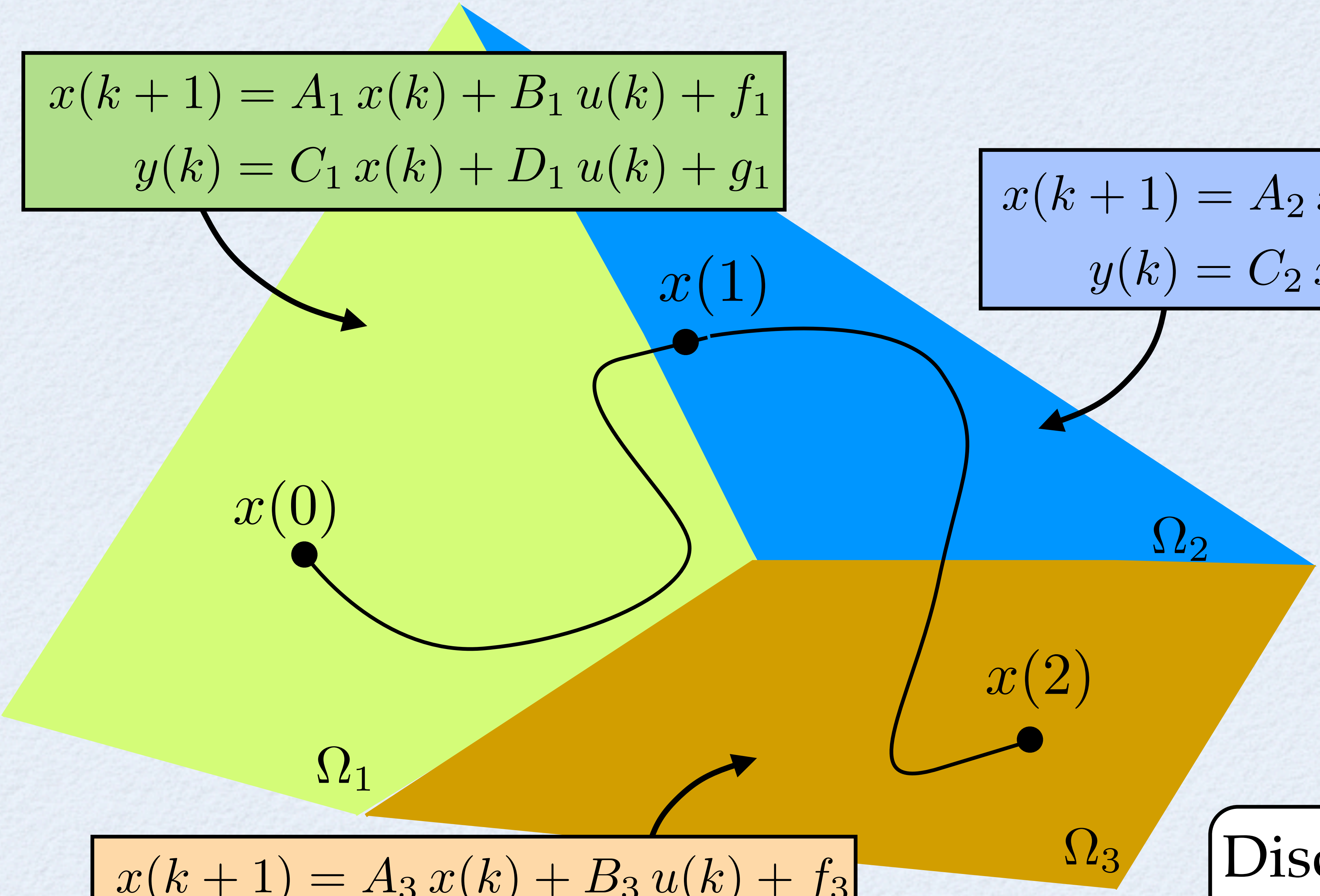


Modeling of hybrid systems

Example of a deterministic PWA model

$$x(k+1) = A_1\, x(k) + B_1\, u(k) + f_1$$
$$y(k) = C_1\, x(k) + D_1\, u(k) + g_1$$

$$x(k+1) = A_2\, x(k) + B_2\, u(k) + f_2$$
$$y(k) = C_2\, x(k) + D_2\, u(k) + g_2$$



$$x(k+1) = A_3\, x(k) + B_3\, u(k) + f_3$$
$$y(k) = C_3\, x(k) + D_3\, u(k) + g_3$$

Discrete mode sequence:

$$\mathbf{i} = [1, 2, 3]$$

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Modeling of hybrid systems

The stochastic PWA model

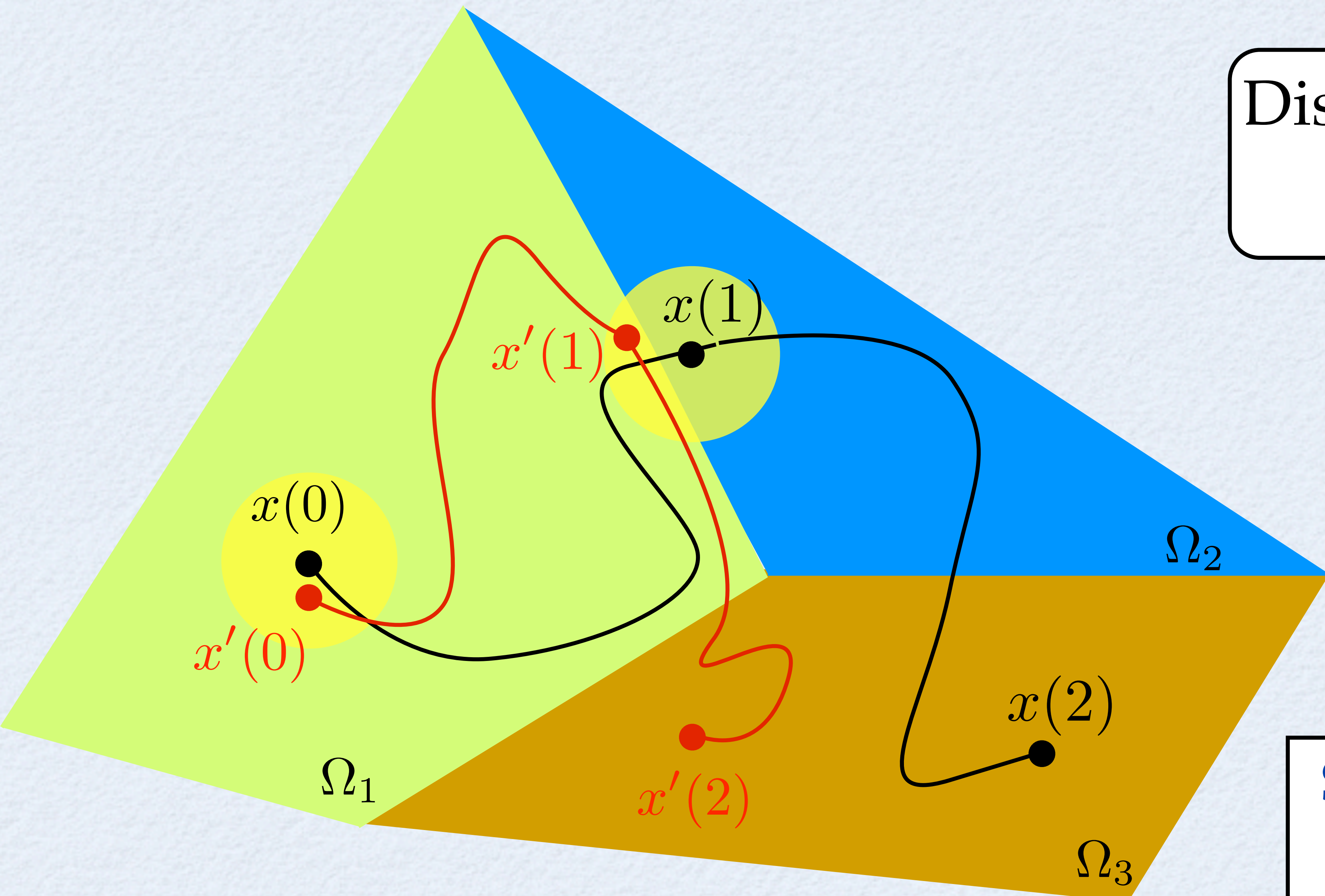
$$\begin{aligned} x(k+1) &= A_{i(k)}\, x(k) + B_{i(k)}\, u(k) + W_{i(k)} w(k) + f_{i(k)} \\ y(k) &= C_{i(k)}\, x(k) + D_{i(k)}\, u(k) + g_{i(k)} + v(k) \\ \Omega_i &\triangleq \left\{ \begin{bmatrix} x(k) \\ u(k) \\ w(k) \end{bmatrix} : S_i x(k) + R_i u(k) + Q_i w(k) \leq T_i \right\} \end{aligned}$$

Mutual impact of the disturbances:

Uncertainty in the continuous state ⟷ Uncertainty in the discrete mode

Modeling of hybrid systems

Analysis of a stochastic PWA model



Discrete mode sequence:

$$\mathbf{j} = [1, 1, 3]$$

Slight state deviation can cause the system to evolve with a different mode sequence.

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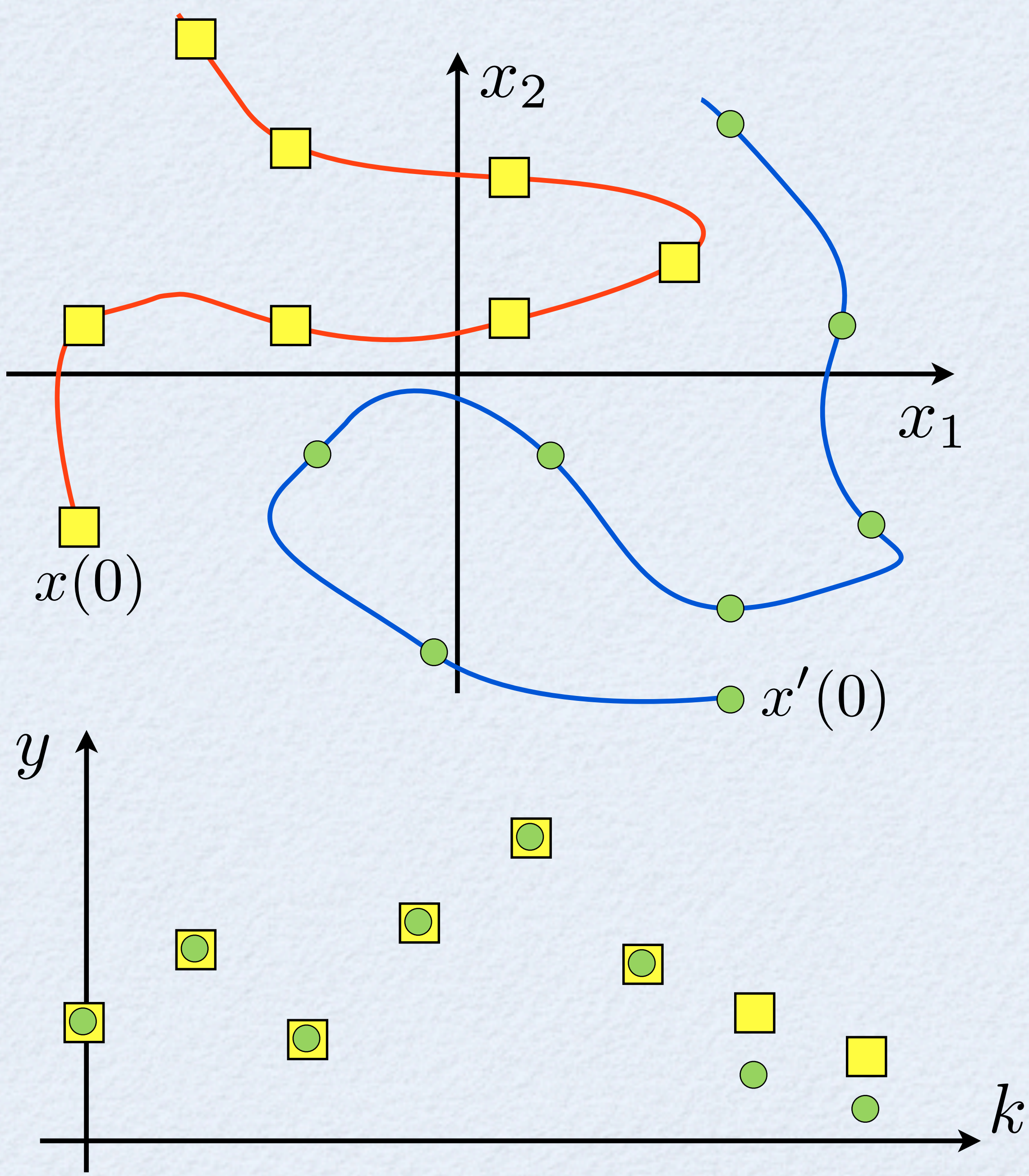
Part II

ESTIMATION OF STOCHASTIC HYBRID SYSTEMS

Estimation of stochastic hybrid systems

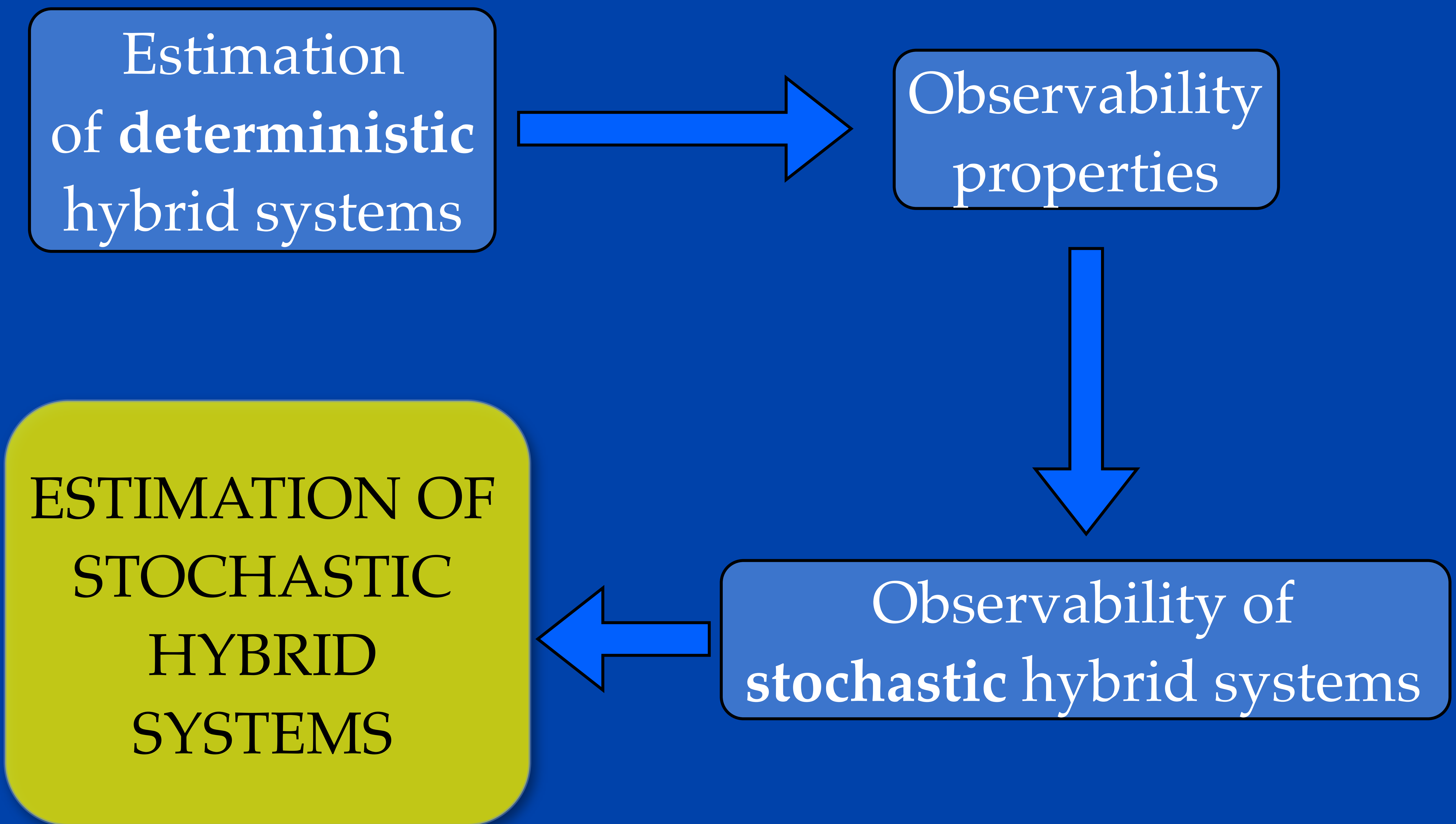
Estimation $\longrightarrow$ Observability $\equiv$ Injectivity



A given output sequence may be produced by more than one trajectory of the system

Observability is not a global property for general hybrid systems !

Estimation of stochastic hybrid systems



The Interacting Multiple Model (IMM)

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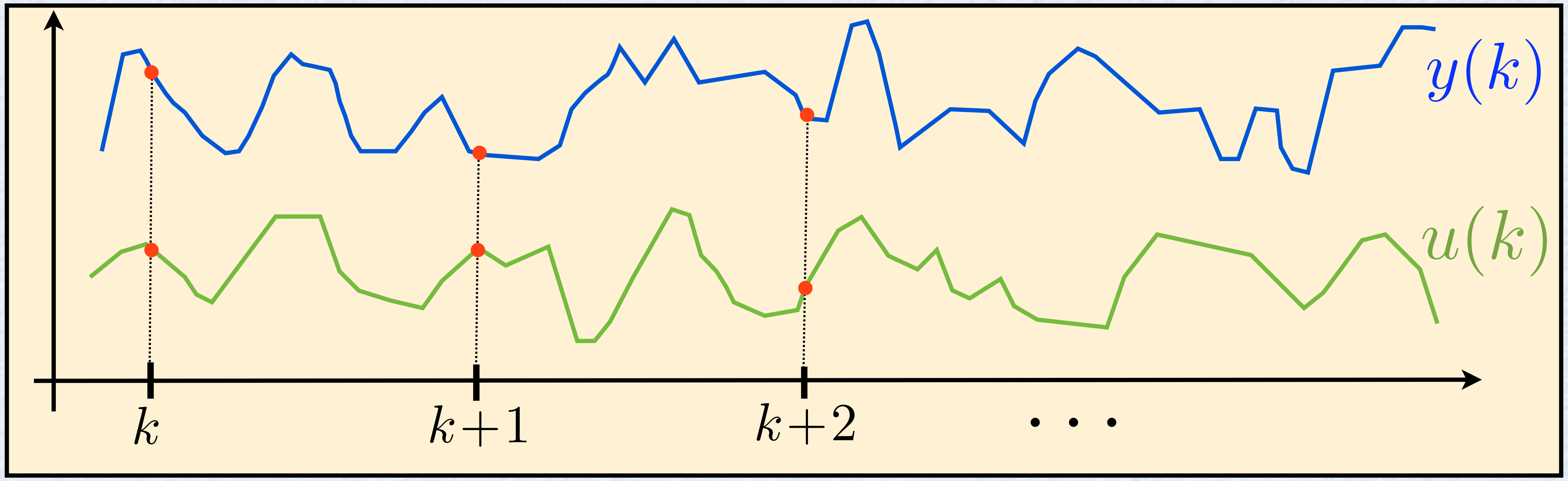
Estimation of deterministic hybrid systems

Problem formulation

Given:



Knowing:

$$x(k+1) = A_{i(k)}\, x(k) + B_{i(k)}\, u(k) + f_{i(k)}$$

$$y(k) = C_{i(k)}\, x(k) + D_{i(k)}\, u(k) + g_{i(k)}$$

$$\Omega_i \triangleq \left\{ \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} : S_i x(t) + R_i u(k) + \leq T_i \right\}$$

Estimate:

$\hat{x}(t)$ – Continuous state evolution

$\hat{i}(k)$ – Discrete mode sequence

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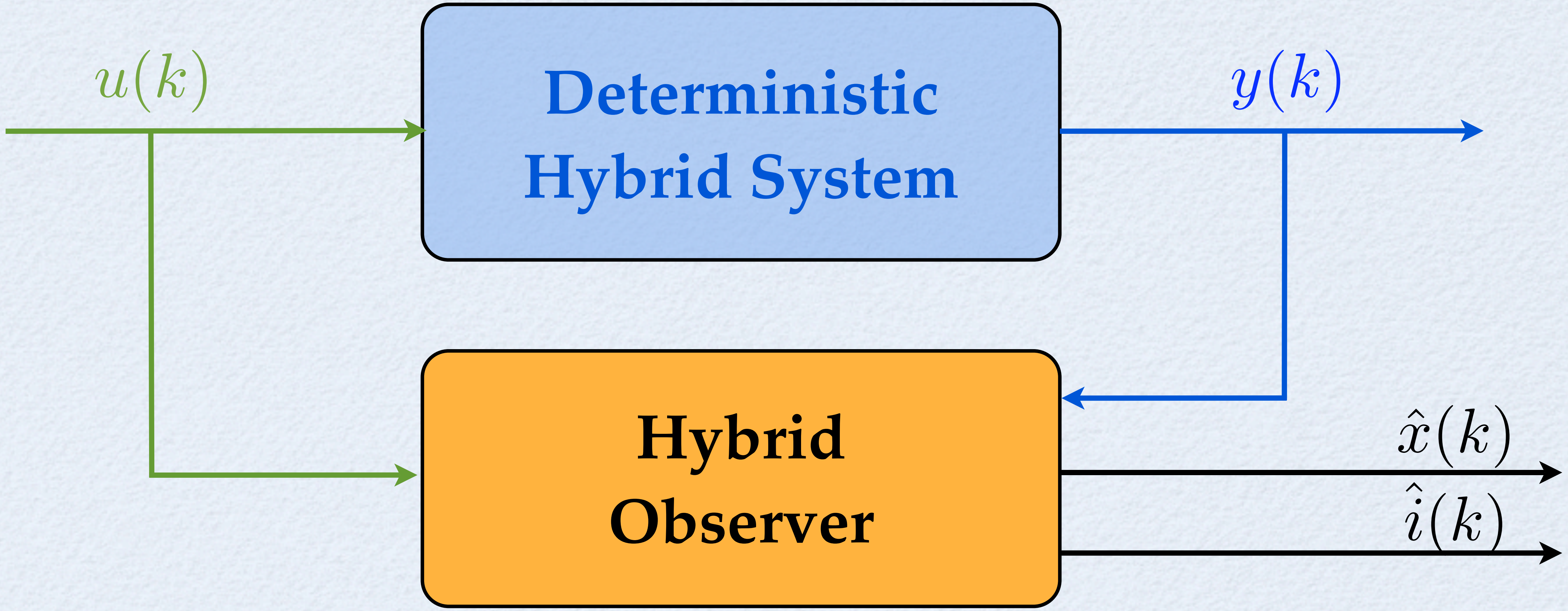
Estimation of deterministic hybrid systems

Hybrid observer design



Methodology:

1) Guarantee Discrete Mode Observability.

2) Guarantee Continuous Mode Observability.

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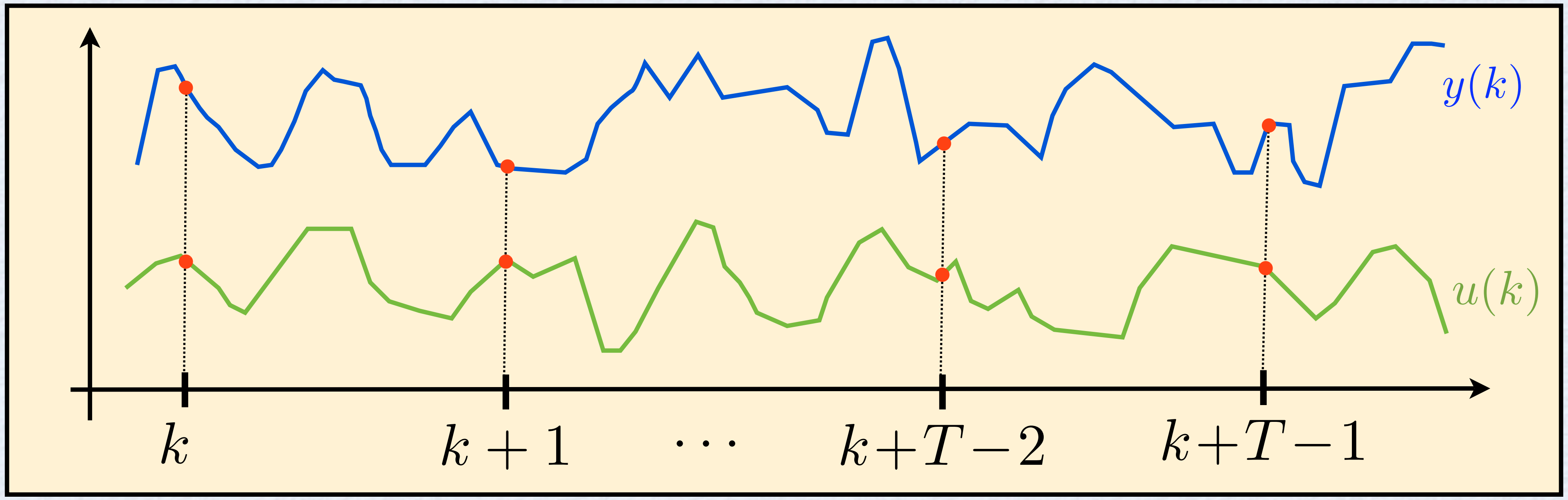
Estimation of deterministic hybrid systems

Dynamics will be analyzed over a *size window* $T : [k, k+T-1]$



Time compressed model over the size window T:

$$
\begin{aligned}
X_T(k) &= \mathbf{A}_{\mathbf{i}_T(k)}\, x(k) + \mathbf{B}_{\mathbf{i}_T(k)}\, U_T(k) + \mathbf{f}_{\mathbf{i}_T(k)} \\
Y_T(k) &= \mathbf{C}_{\mathbf{i}_T(k)}\, x(k) + \mathbf{D}_{\mathbf{i}_T(k)}\, U_T(k) + \mathbf{g}_{\mathbf{i}_T(k)} \\
\mathbf{\Omega}_{\mathbf{i}_T} &\triangleq \left\{ \begin{bmatrix} x(k) \\ U_T(k) \end{bmatrix} : \mathbf{S}_{\mathbf{i}_T}\, x(k) + \mathbf{R}_{\mathbf{i}_T}\, U_T(k) \leq \mathbf{T}_{\mathbf{i}_T} \right\}
\end{aligned}
$$

Estimation of deterministic hybrid systems

Matrix definitions...

$$\mathbf{A}_{\mathbf{i}_T} = \begin{bmatrix} I_{n_x} \\ A_{i_0} \\ A_{i_1}A_{i_0} \\ \vdots \\ A_{i_{T-2}}\dots A_{i_1}A_{i_0} \end{bmatrix}$$

$$\mathbf{B}_{\mathbf{i}_T} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ B_{i_0} & 0 & \cdots & 0 & 0 \\ A_{i_1}B_{i_0} & B_{i_1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{i_{T-2}}\dots A_{i_1}B_{i_0} & A_{i_{T-2}}\dots A_{i_2}B_{i_1} & \cdots & B_{i_{T-2}} & 0 \end{bmatrix}$$

$$\mathbf{C}_{\mathbf{i}_T} = \begin{bmatrix} C_{i_0} \\ C_{i_1}A_{i_0} \\ C_{i_2}A_{i_1}A_{i_0} \\ \vdots \\ C_{i_{T-1}}A_{i_{T-2}}\dots A_{i_1}A_{i_0} \end{bmatrix}$$

$$\mathbf{D}_{\mathbf{i}_T} = \begin{bmatrix} D_{i_1} & 0 & \cdots & 0 & 0 \\ C_{i_1}B_{i_0} & D_{i_1} & \cdots & 0 & 0 \\ C_{i_2}A_{i_1}B_{i_0} & C_{i_2}B_{i_1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ C_{i_{T-1}}A_{i_{T-2}}\dots A_{i_1}B_{i_0} & C_{i_{T-1}}A_{i_{T-2}}\dots A_{i_2}B_{i_1} & \cdots & C_{i_{T-1}}B_{i_{T-2}} & D_{i_{T-1}} \end{bmatrix}$$

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Observability of deterministic hybrid systems

Discrete Mode Observability

A PWA system is **Mode Observable** iff for any pair of feasible hybrid trajectories:

$$\left(x_{\mathbf{i}}, U_T, \mathbf{i}_T\right) \quad , \quad \left(x_{\mathbf{j}}, U_T, \mathbf{j}_T\right)$$

the following holds:

$$\mathbf{i}_T \neq \mathbf{j}_T \Rightarrow Y\left(x_{\mathbf{i}}, U_T, 0, 0, \mathbf{i}_T\right) \neq Y\left(x_{\mathbf{j}}, U_T, 0, 0, \mathbf{j}_T\right)$$

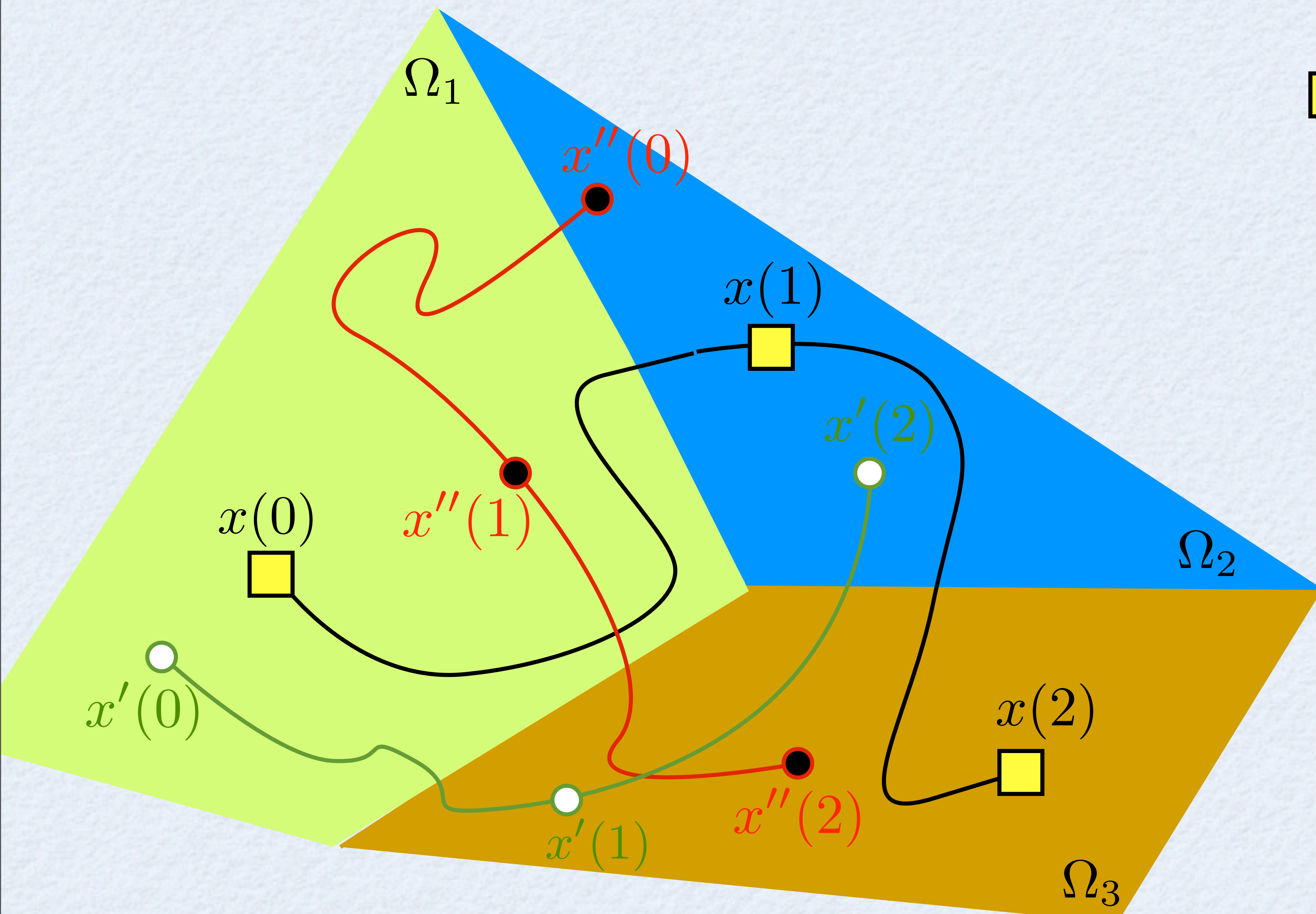
i.e., there is no overlapping between both output feasibility polytopes:

$$\mathcal{Y}_{\mathbf{i}_T} \quad , \quad \mathcal{Y}_{\mathbf{j}_T}$$

Discrete Mode Observability	$\equiv$	Separability of the outputs for different discrete mode sequences

Observability of deterministic hybrid systems

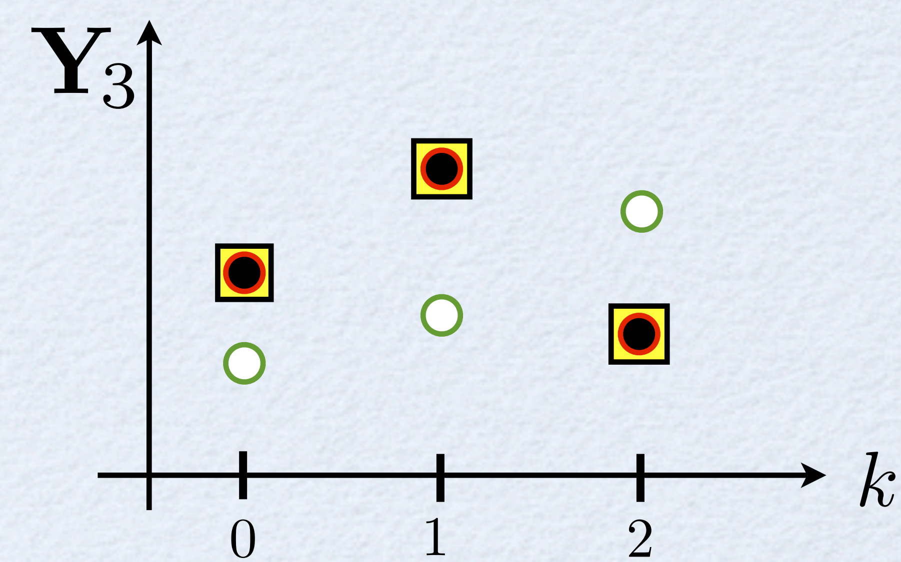
Discrete mode observability: *injectivity in the mode*



$\mathbf{j}_3 = [1\ 2\ 3]$

$\mathbf{i}_3 = [1\ 3\ 2]$

$\mathbf{k}_3 = [2\ 1\ 3]$



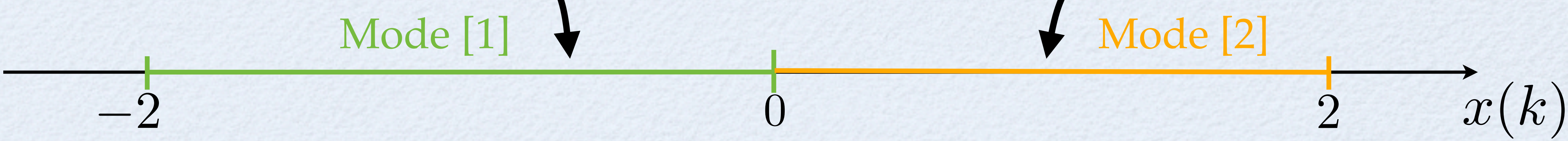
Discrete mode sequences $\mathbf{j}_3$ and $\mathbf{k}_3$ are **not observable** from $y(k)$.

Observability of deterministic hybrid systems

Example: deterministic PWA system with 2 modes

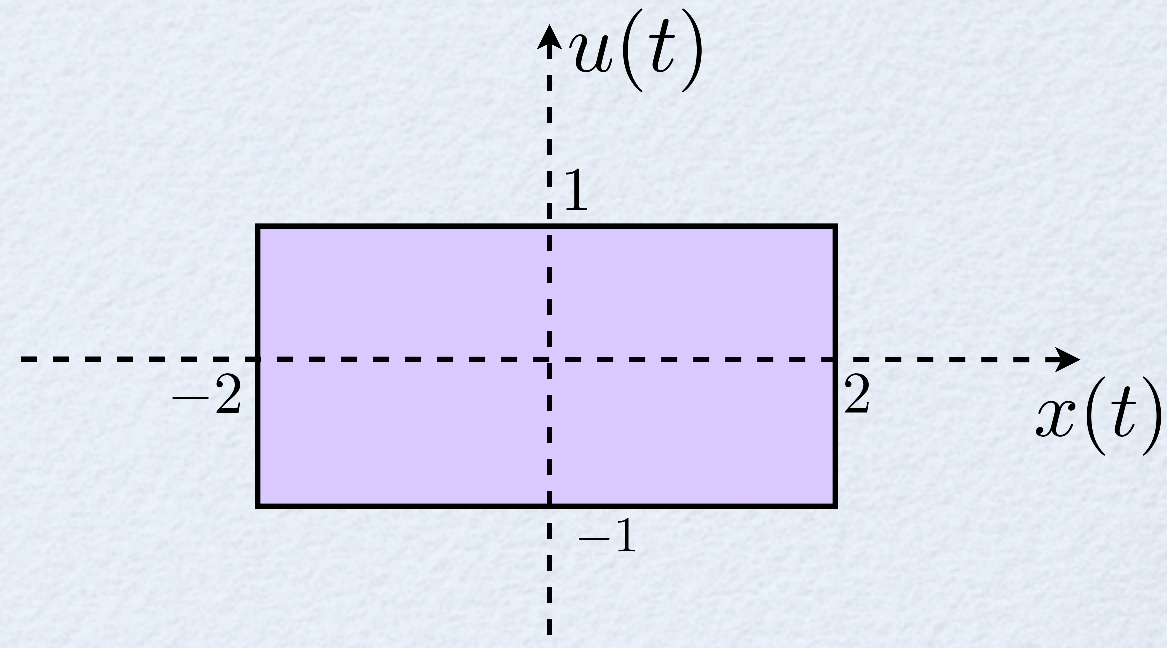
$$\begin{aligned} x(k+1) &= -0.5\,x(k) + u(k) \\ y(k) &= x(k) \end{aligned}$$

$$\begin{aligned} x(k+1) &= -x(k) + u(k) \\ y(k) &= x(k) - 0.5 \end{aligned}$$



defined in the polytopic region:

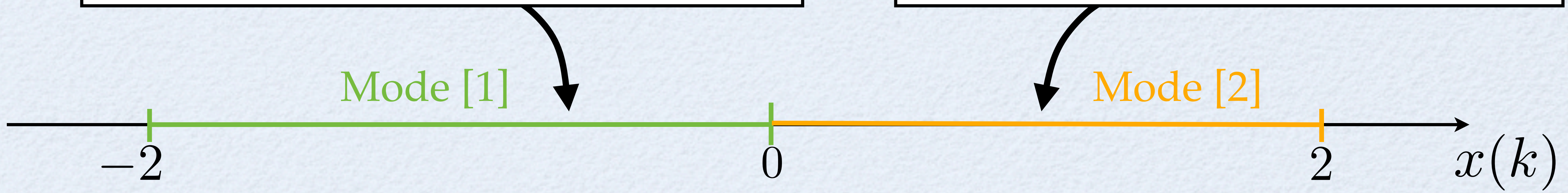
$$\begin{aligned} x(k) &\in \mathbb{X} \triangleq [-2\,,\ 2] \\ u(k) &\in \mathbb{U} \triangleq [-1\,,\ 1] \end{aligned}$$



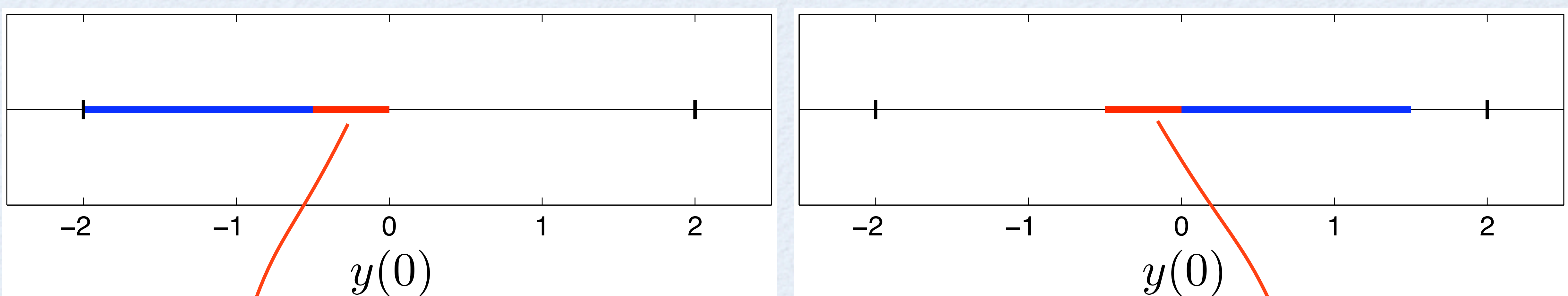
Observability of deterministic hybrid systems

$$\begin{aligned} x(k+1) &= -0.5\,x(k) + u(k) \\ y(k) &= x(k) \end{aligned}$$

$$\begin{aligned} x(k+1) &= -x(k) + u(k) \\ y(k) &= x(k) - 0.5 \end{aligned}$$



Output feasibility polytopes for T = 1:

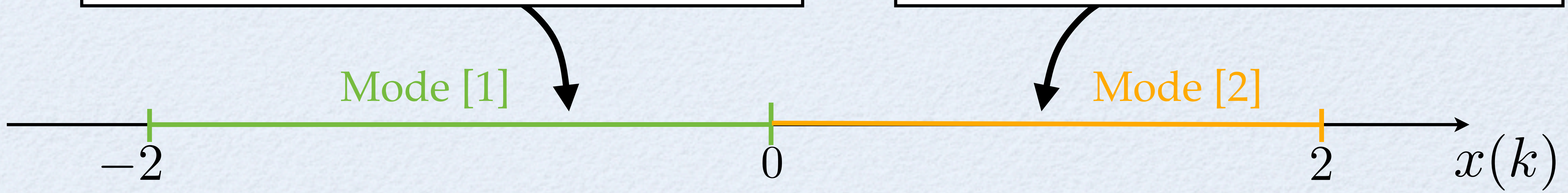


The mode is **not observable** from the output

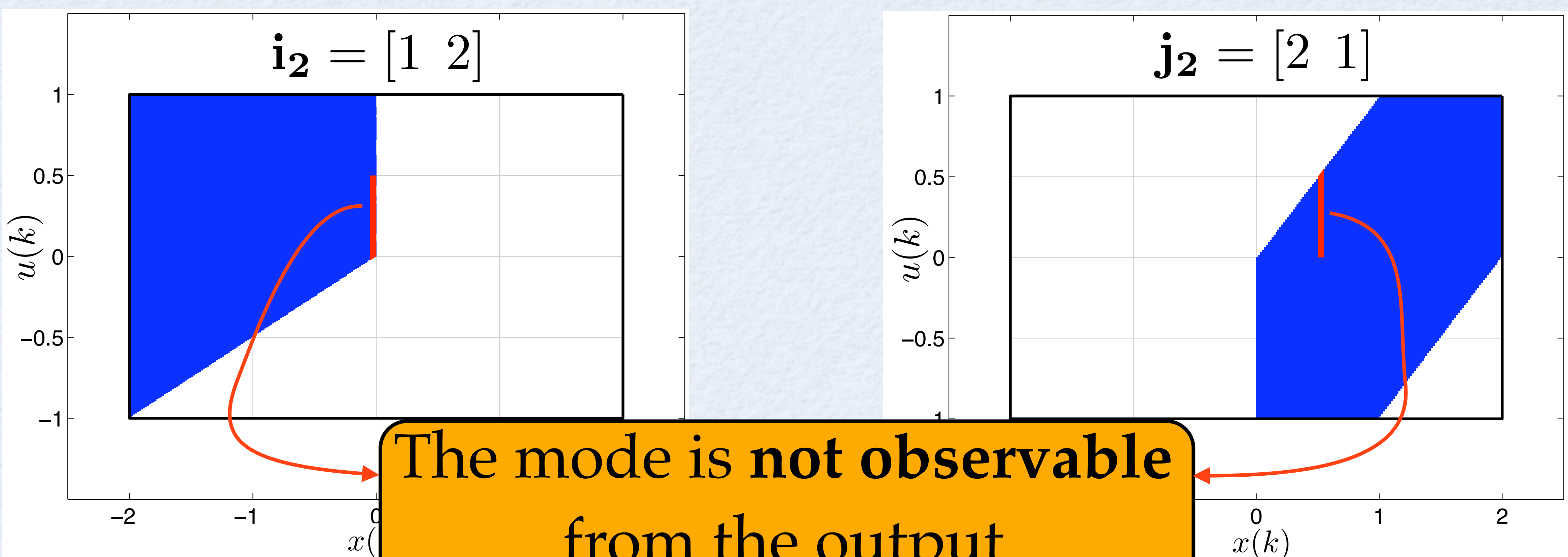
Observability of deterministic hybrid systems

$$\begin{aligned} x(k+1) &= -0.5\,x(k) + u(k) \\ y(k) &= x(k) \end{aligned}$$

$$\begin{aligned} x(k+1) &= -x(k) + u(k) \\ y(k) &= x(k) - 0.5 \end{aligned}$$



Overlapping output regions for $\mathbf{T=2}$:



The mode is **not observable** from the output

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Observability of deterministic hybrid systems

Continuous State Observability

A PWA system is **Pathwise Observable** iff there exists a finite horizon T such that all feasible discrete mode sequences $\{i(0), \ldots, i(T-1)\}$ are observable, i.e.:

$$\text{rank}(\mathbf{C}_{\mathbf{i}_T}) = \text{rank}\left(\begin{bmatrix} C_{i_0} \\ C_{i_1} A_{i_0} \\ C_{i_2} A_{i_1} A_{i_0} \\ \vdots \\ C_{i_{T-1}} A_{i_{T-2}} \ldots A_{i_1} A_{i_0} \end{bmatrix}\right) = n$$

The smallest value for T, T_{PWO}, is the index of the PWO.

The observations: $\{y(0), \ldots, y(T-1)\}$ → Uniquely determine: x_0 for every admissible DMS.

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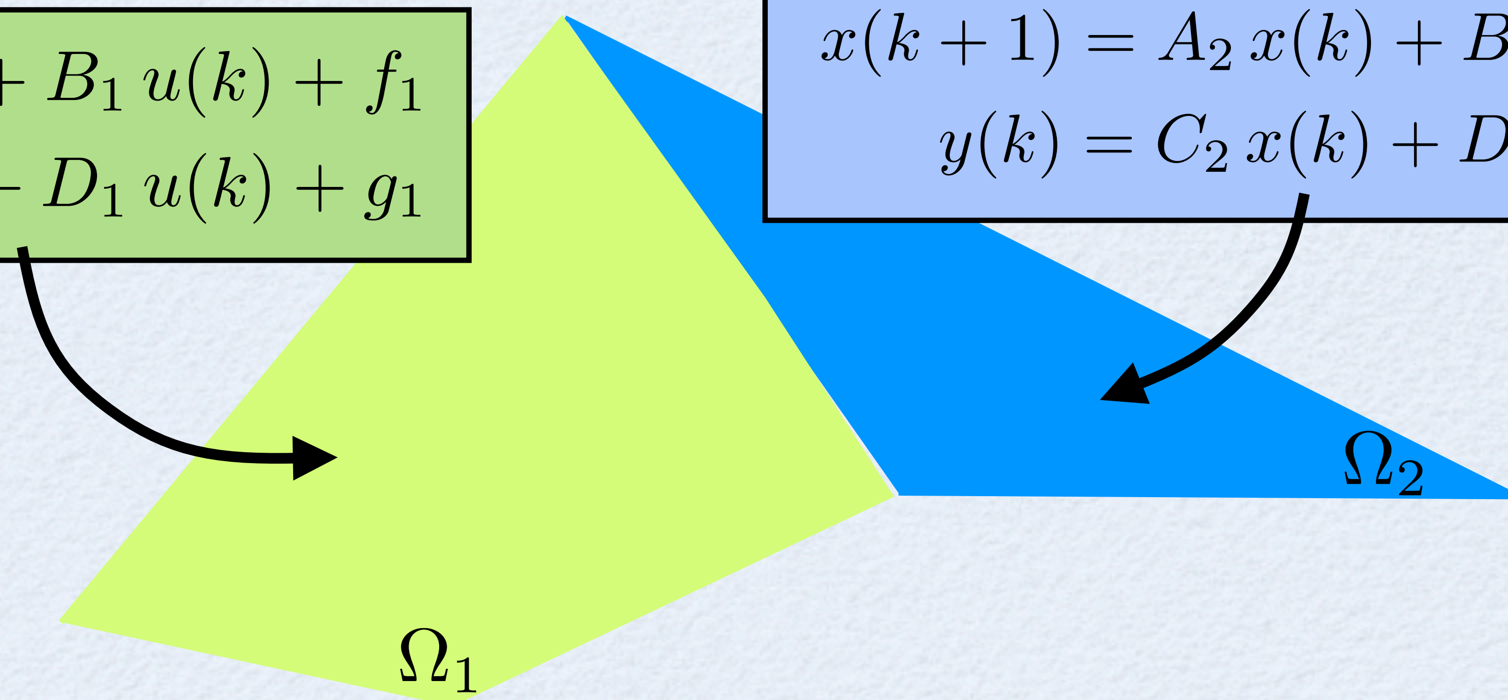
Observability of deterministic hybrid systems

Example: PWA system with $n = 3$, and 2 modes: $i = \{1, 2\}$

$$\begin{aligned} x(k+1) &= A_1\, x(k) + B_1\, u(k) + f_1 \\ y(k) &= C_1\, x(k) + D_1\, u(k) + g_1 \end{aligned}$$

$$\begin{aligned} x(k+1) &= A_2\, x(k) + B_2\, u(k) + f_2 \\ y(k) &= C_2\, x(k) + D_2\, u(k) + g_2 \end{aligned}$$



$$\operatorname{rank}\left(\begin{bmatrix} C_1 \\ C_1 A_1 \\ C_1 A_1^2 \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} C_1 \\ C_2 A_1 \\ C_1 A_2 A_1 \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} C_2 \\ C_1 A_2 \\ C_2 A_2 A_1 \\ C_2 A_2 A_2 A_1 \end{bmatrix}\right) = \cdots = 3$$

[1 1 1] [1 2 1] [2 1 2 2] [...]

How to find the smallest value for T that is enough to prove pathwise observability?

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Observability of deterministic hybrid systems

Continuous State Observability

A PWA system is **State Observable** iff:

➡ System is **Pathwise Observable** with index PWO $(T = T_{PWO})$:

$$\Downarrow$$

Unique determination of x_0 *when the DMS of length* T_{PWO} *is known.*

➡ System is **Mode Observable** at T_{PWO}:

$$\Downarrow$$

Unique DMS $\mathbf{i}_{T_{PWO}}$ *produces the measured outputs.*

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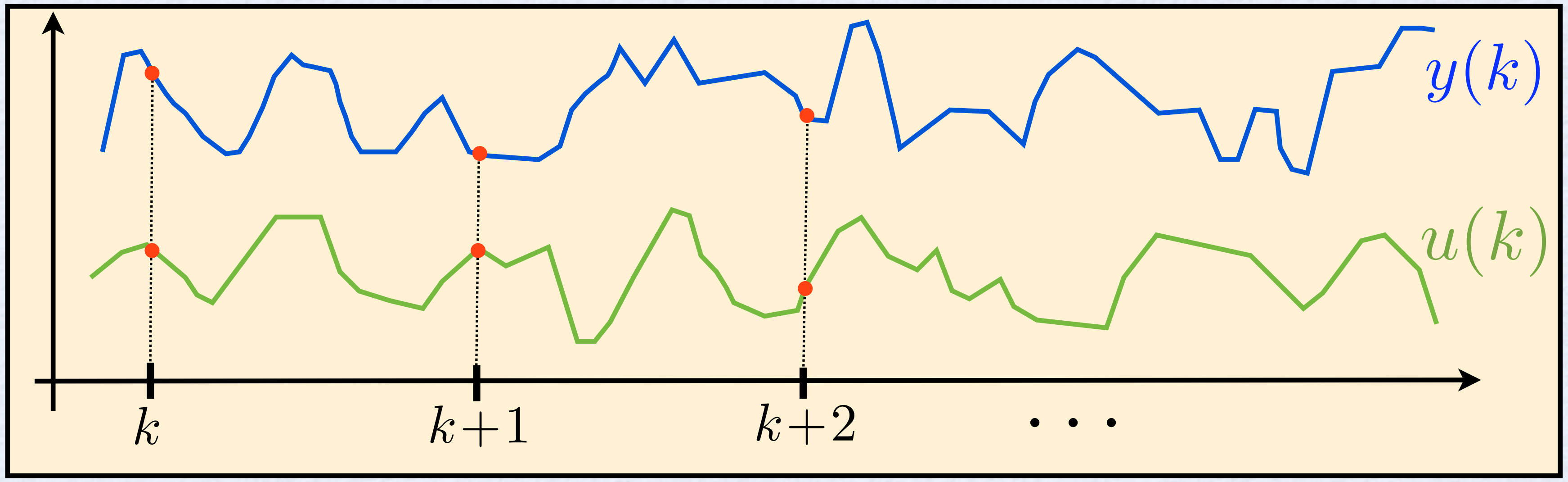
Observability of stochastic hybrid systems

Problem formulation

Given:



Knowing:

$$
\begin{aligned}
x(k+1) &= A_{i(k)}\,x(k) + B_{i(k)}\,u(k) + W_{i(k)}w(k) + f_{i(k)} \\
y(k) &= C_{i(k)}\,x(k) + D_{i(k)}\,u(k) + g_{i(k)} + v(k) \\
\Omega_i &\triangleq \left\{ \begin{bmatrix} x(k) \\ u(k) \\ w(k) \end{bmatrix} : S_i x(k) + R_i u(k) + Q_i w(k) \le T_i \right\}
\end{aligned}
$$

Estimate:

$\hat{x}(k)$ – Continuous state evolution

$\hat{i}(k)$ – Discrete mode sequence

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Observability of stochastic hybrid systems

The *time compressed* stochastic model

$$
\begin{aligned}
X_T(k) &= \mathbf{A}_{\mathbf{i}_T(k)}\, x(k) + \mathbf{B}_{\mathbf{i}_T(k)} U_T(k) + \mathbf{W}_{\mathbf{i}_T(k)} W_T(k) + \mathbf{f}_{\mathbf{i}_T(k)} \\
Y_T(k) &= \mathbf{C}_{\mathbf{i}_T(k)} x(k) + \mathbf{D}_{\mathbf{i}_T(k)} U_T(k) + \mathbf{g}_{\mathbf{i}_T(k)} + \mathbf{L}_{\mathbf{i}_T(k)} W_T(k) + V_T(k) \\
\boldsymbol{\Omega}_{\mathbf{i}_T} &\triangleq \left\{ \begin{bmatrix} x(k) \\ U_T(k) \\ W_T(k)(k) \end{bmatrix} : \mathbf{S}_{\mathbf{i}_T} x(k) + \mathbf{R}_{\mathbf{i}_T} U_T(k) + \mathbf{Q}_{\mathbf{i}_T} W_T(k) \leq \mathbf{T}_{\mathbf{i}_T} \right\}
\end{aligned}
$$

1) The hybrid trajectories $X_T(k)$ are characterized in probability.

2) Several DMS $\mathbf{i}_T$ are candidate to have produced a given measurement sequence, although some with a higher probability than others.

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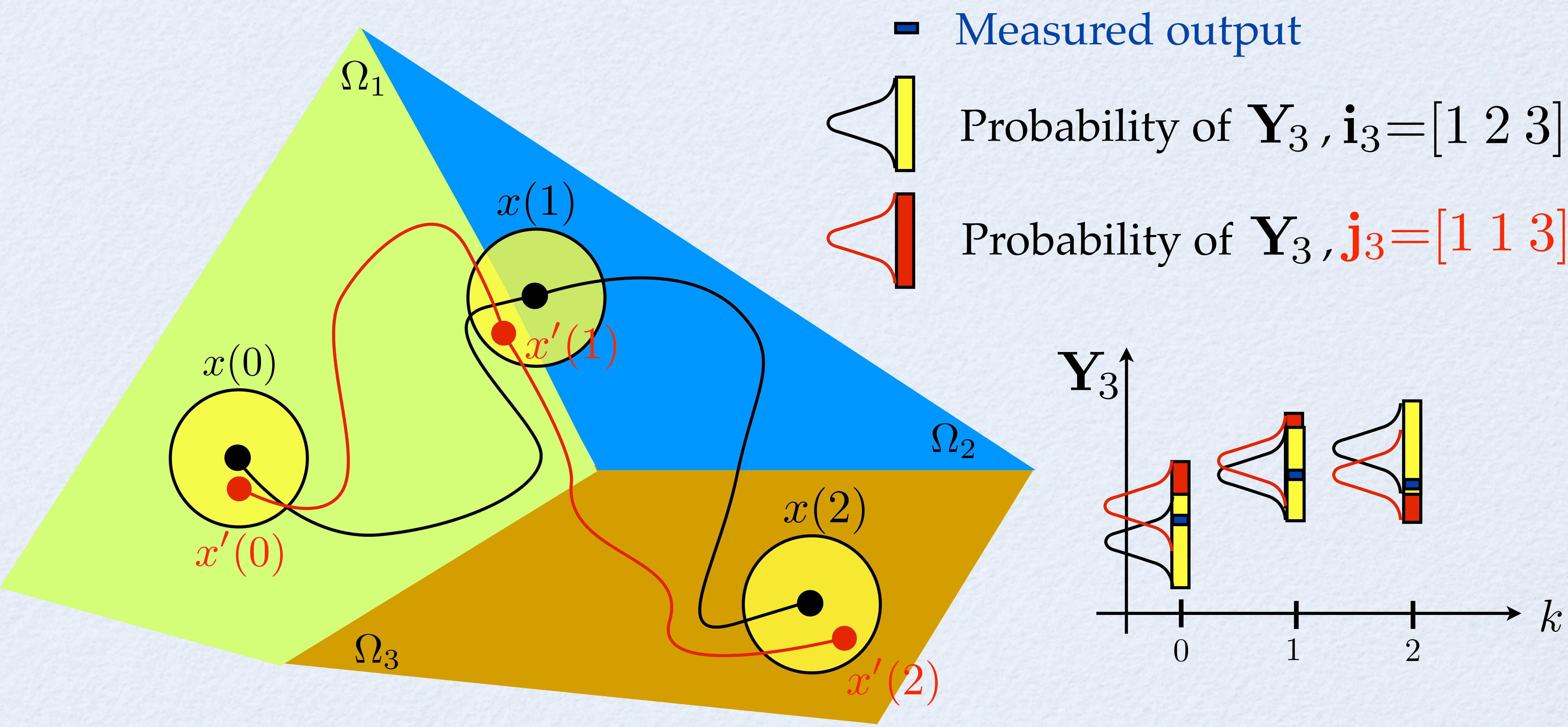
Observability of stochastic hybrid systems

Discrete Mode Observability in Probability



Which DMS $\mathbf{i}_3$ or $\mathbf{j}_3$ is more likely to have produced $\mathbf{Y}_3$?

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Observability of stochastic hybrid systems

Discrete Mode Observability in Probability

For a given fixed input and output data sequences:

$$(U_T, Y_T)$$

Find the DMS with the highest probability of matching (U_T, Y_T)

$$\mathbf{i}^*_T(Y_T, U_T, \mathcal{J}_T)$$

fixed (Y_T, U_T)

Set of admissible DMS with length T ($\mathcal{J}_T$)

Mode Observability is given in probability!
No longer a YES/NO answer!

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Observability of stochastic hybrid systems

Discrete Mode Observability in Probability

DMS least squares estimator:

$$\hat{\mathbf{i}}_T^*\big(Y_T, U_T, \mathcal{J}_T\big) = \arg \min_{\mathbf{j}_T \in \mathcal{J}_T} \|Y_T - \tilde{\mathcal{Y}}_{\mathbf{j}_T}(U_T)\|^2$$

DMS maximum likelihood estimator:

$$\hat{\mathbf{i}}_T^*\big(Y_T, U_T, \mathcal{J}_T\big) = \arg \max_{\mathbf{j}_T \in \mathcal{J}_T} \Pr\big(Y_T \in \tilde{\mathcal{Y}}_{\mathbf{j}_T}(U_T)\big)$$

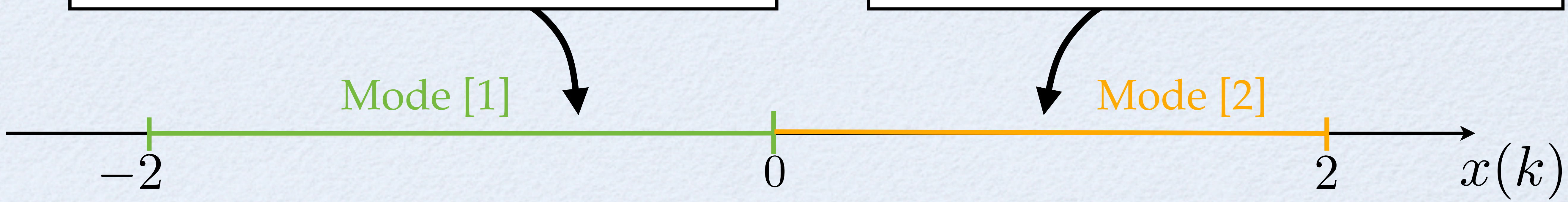
System is **Mode Observable** with at least probability P_{MO}

Observability of stochastic hybrid systems

Example: stochastic PWA system with 2 modes

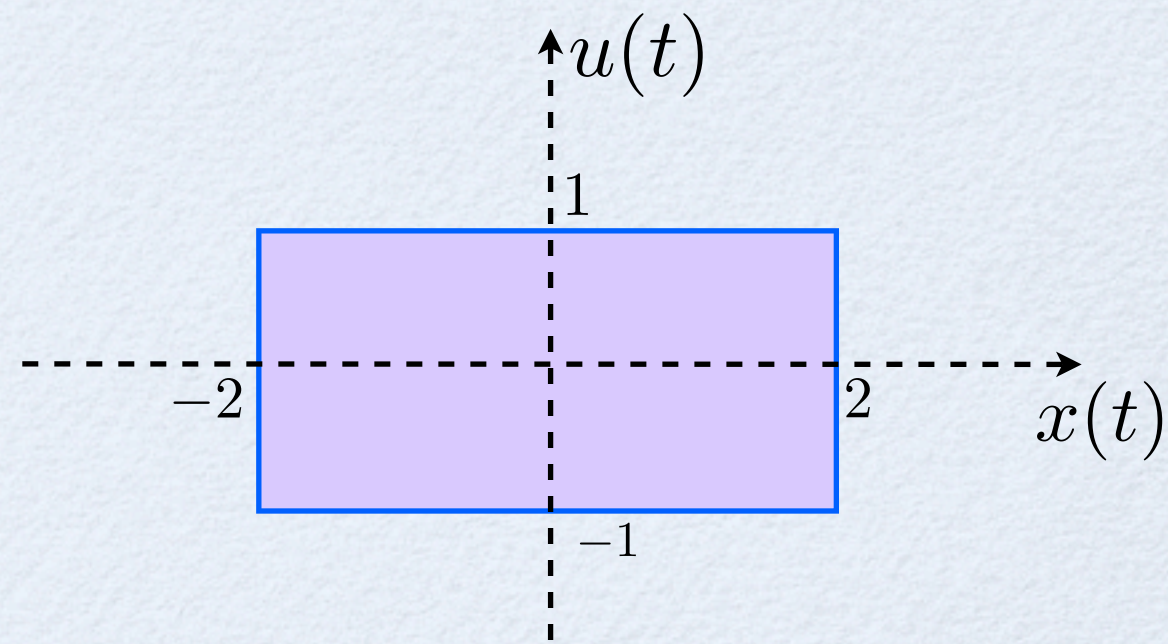
$$\begin{aligned} x(k+1) &= -0.5\,x(k) + u(k) \\ y(k) &= x(k) + v(k) \\ v(k) &\in \mathbb{V}_1 \triangleq [-0.2\ ,\ 0.2] \end{aligned}$$

$$\begin{aligned} x(k+1) &= -x(k) + u(k) \\ y(k) &= x(k) - 0.5 + v(k) \\ v(k) &\in \mathbb{V}_2 \triangleq [-0.5\ ,\ 0.5] \end{aligned}$$



defined in the polytopic region:

$$\begin{aligned} x(k) &\in \mathbb{X} \triangleq [-2\ ,\ 2] \\ u(k) &\in \mathbb{U} \triangleq [-1\ ,\ 1] \end{aligned}$$



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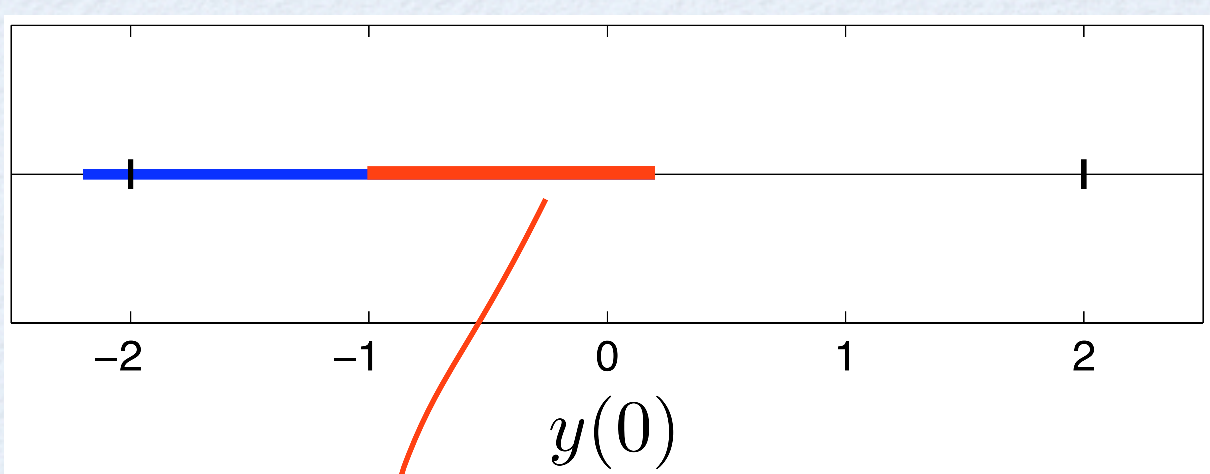
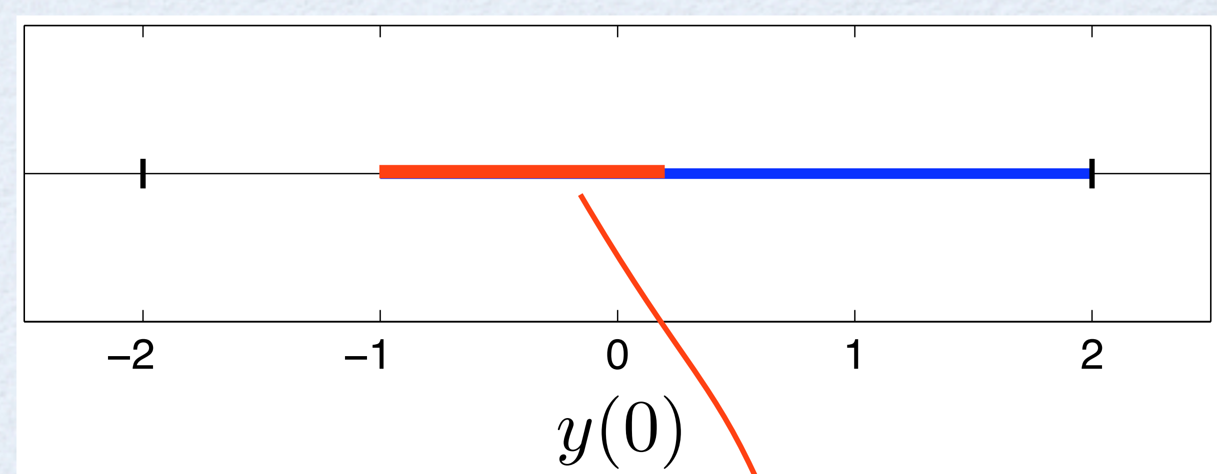
Observability of stochastic hybrid systems

$$\begin{aligned} x(k+1) &= -0.5\,x(k) + u(k) \\ y(k) &= x(k) + v(k) \\ v(k) &\in \mathbb{V}_1 \triangleq [-0.2\ ,\ 0.2] \end{aligned}$$

$$\begin{aligned} x(k+1) &= -x(k) + u(k) \\ y(k) &= x(k) - 0.5 + v(k) \\ v(k) &\in \mathbb{V}_2 \triangleq [-0.5\ ,\ 0.5] \end{aligned}$$

Output feasibility polytopes for T = 1:





The mode is **not observable** from the output

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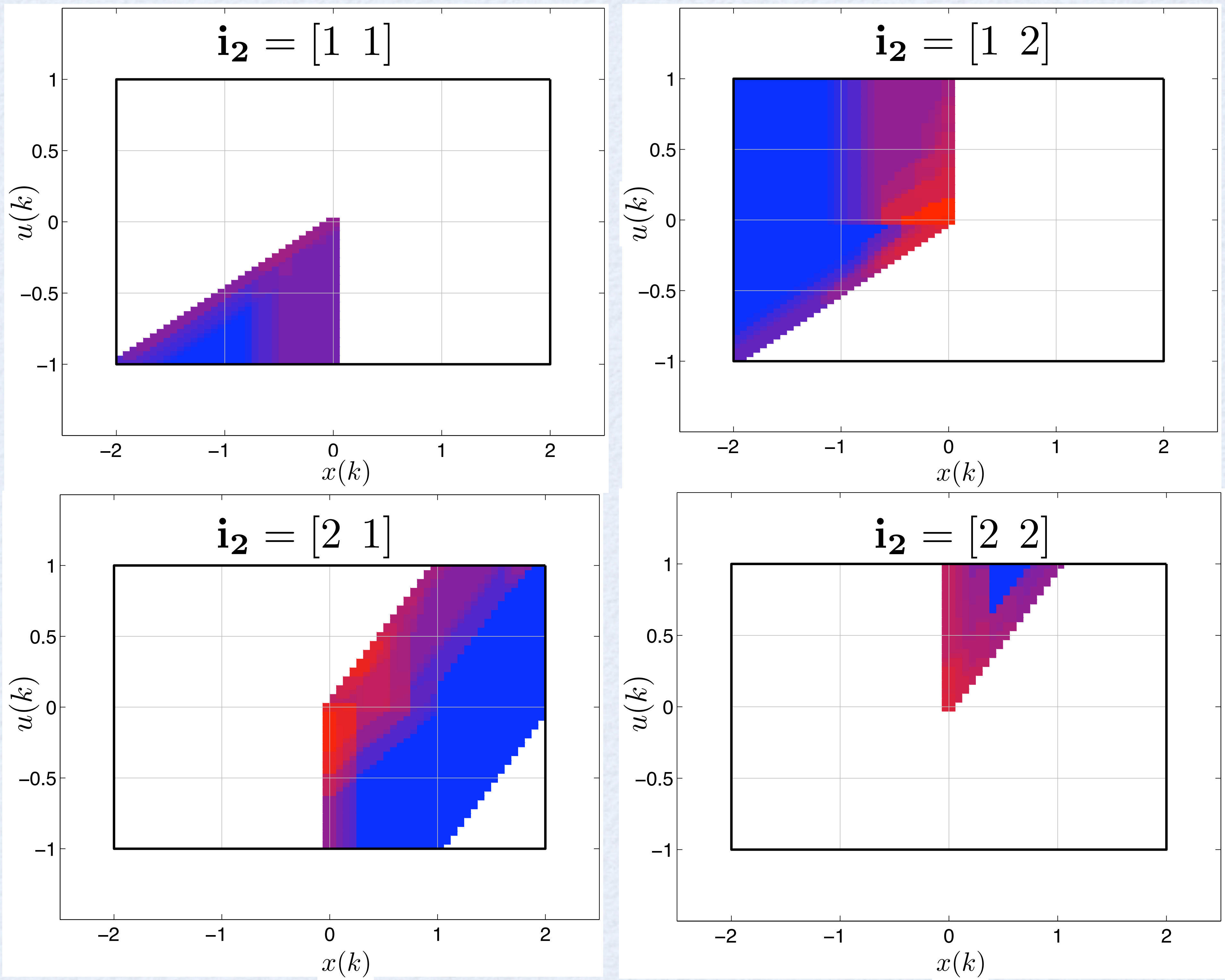
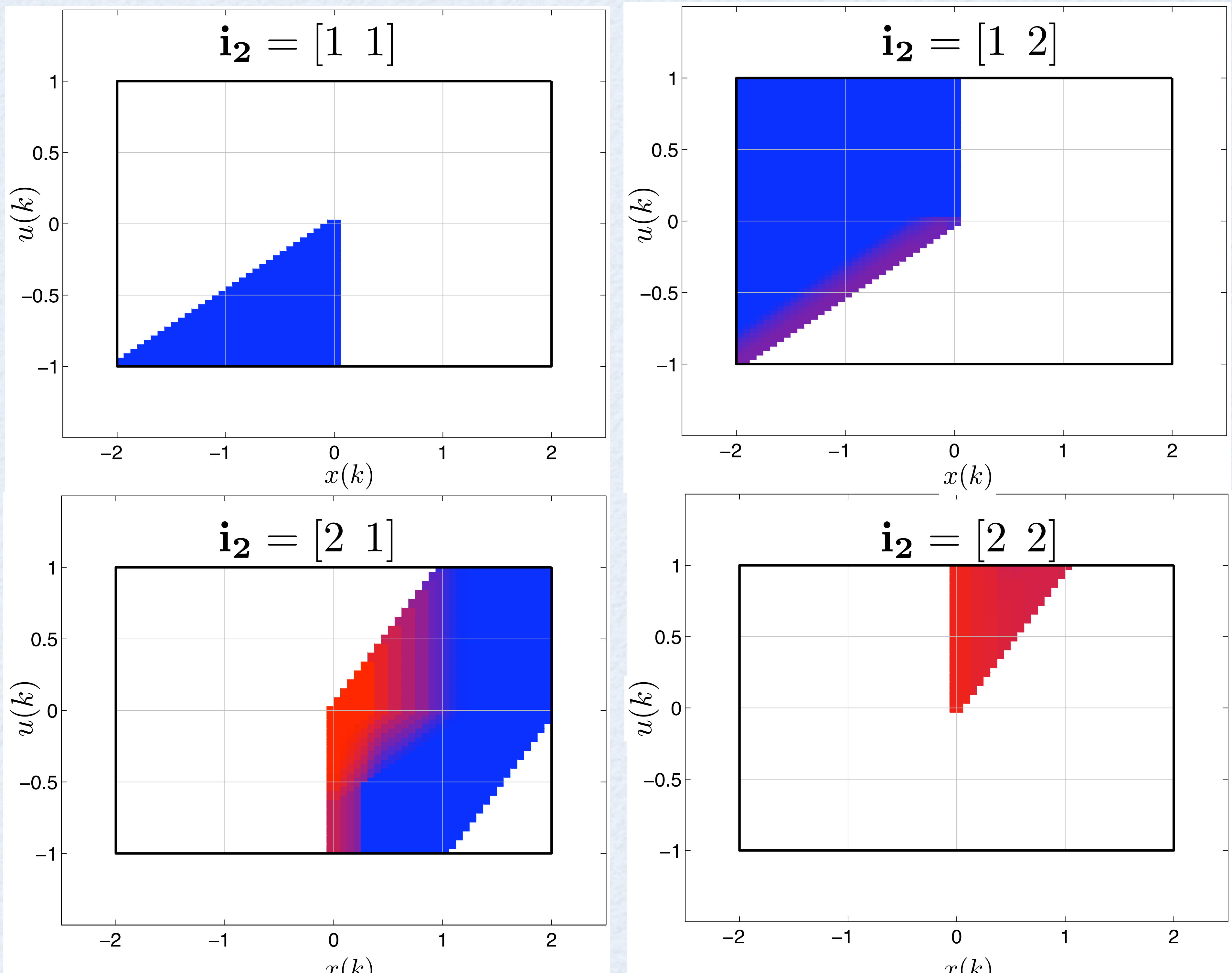
Observability of stochastic hybrid systems

Probability of correct mode estimation (Least Squares Estimator)



Observability of stochastic hybrid systems

Probability of correct mode estimation (Max. Likelihood Estimator)



Observability of stochastic hybrid systems

Continuous State Observability

In order to guarantee **Pathwise Observability**:

The observations: $\{y(0), \ldots, y(T-1)\}$	→	Uniquely determine: x_0 for every admissible DMS.

- The same condition as for the deterministic case.
- But all in probabilistic terms...

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Observability of stochastic hybrid systems

Continuous State Observability in Probability

A PWA system is **State Observable** with a given probability iff:

➡ System is **Pathwise Observable** with index PWO $(T = T_{PWO})$:

$$\Downarrow$$

Unique determination of x_0 in a least squares sense when the DMS of length T_{PWO} is known.

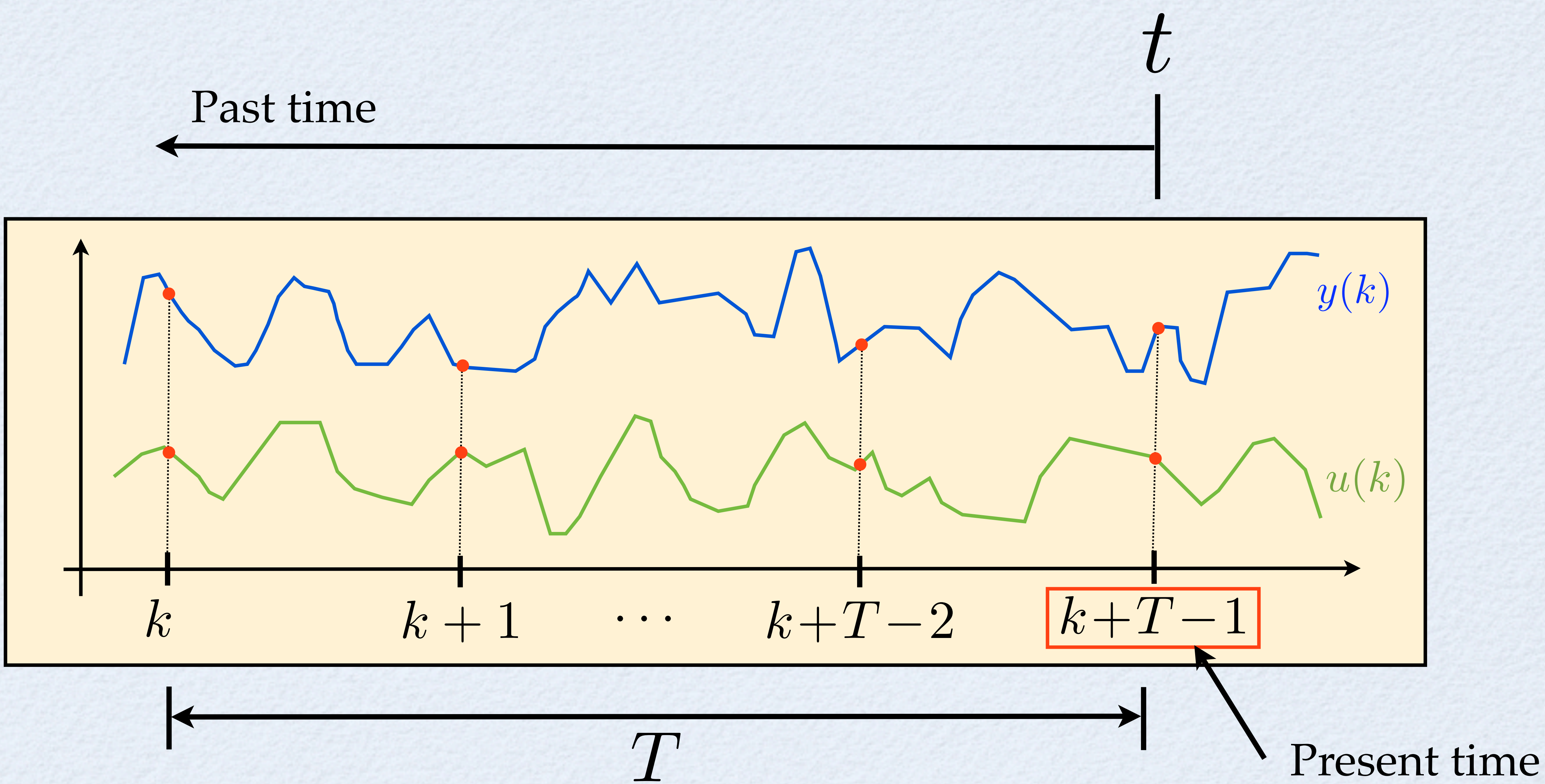
➡ System is **Mode Observable** with probability P_{MO}:

$$\Downarrow$$

$\hat{\mathbf{i}}^*_{T_{PWO}}$ *has the highest probability of being the correct mode estimation.*

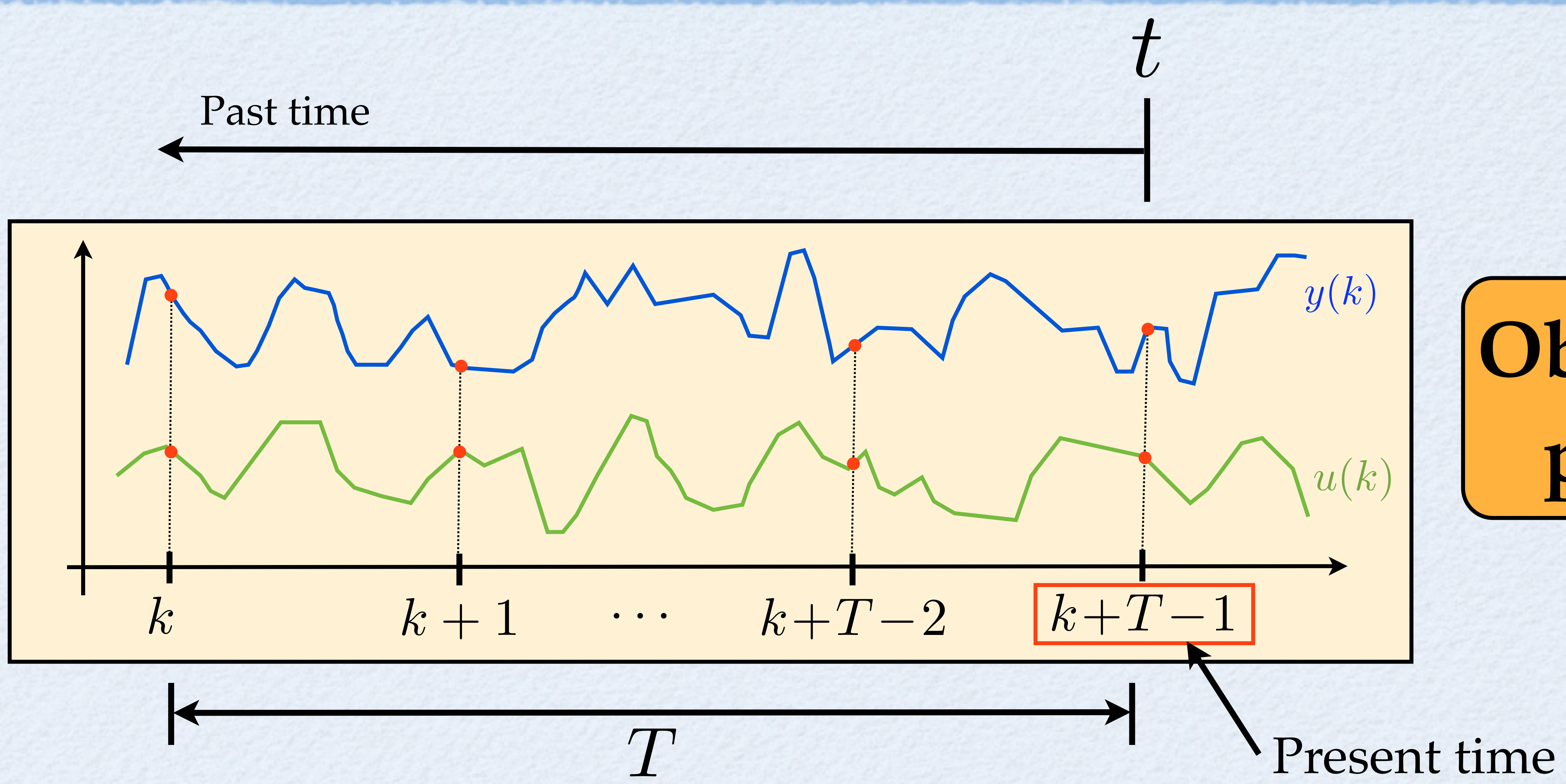
Estimation of stochastic hybrid systems

Problem formulation



Objective: estimate $\hat{x}(t)$ and the current discrete mode $\hat{\mathbf{i}}(t)$

Estimation of stochastic hybrid systems



Observability properties

Reconstruct from $k \to k+T+1$

Discrete mode sequence: $\hat{\mathbf{i}}_T$

Continuous time sequences:

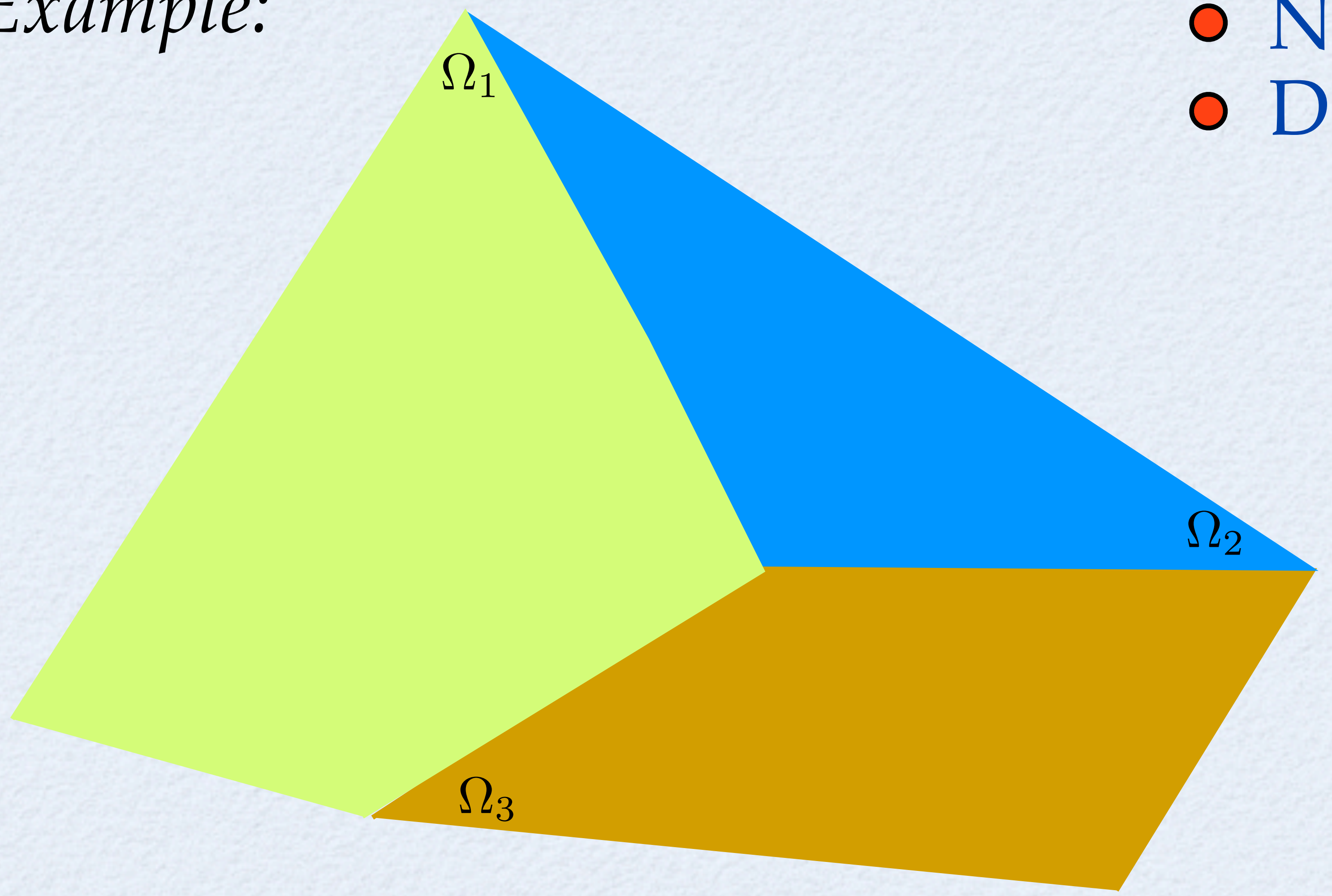
$\hat{x}_{\mathbf{i}_T}(k|t)\ ,\ \hat{w}_{\mathbf{i}_T}(k|t)\ ,\ \hat{v}_{\mathbf{i}_T}(k|t)$

Estimation of stochastic hybrid systems

General Estimation Problem

Find from all possible DMS $\mathbf{j}_T$:

Example:



- Number of modes: 3
- Data sequence dimension: 4

$\Downarrow$

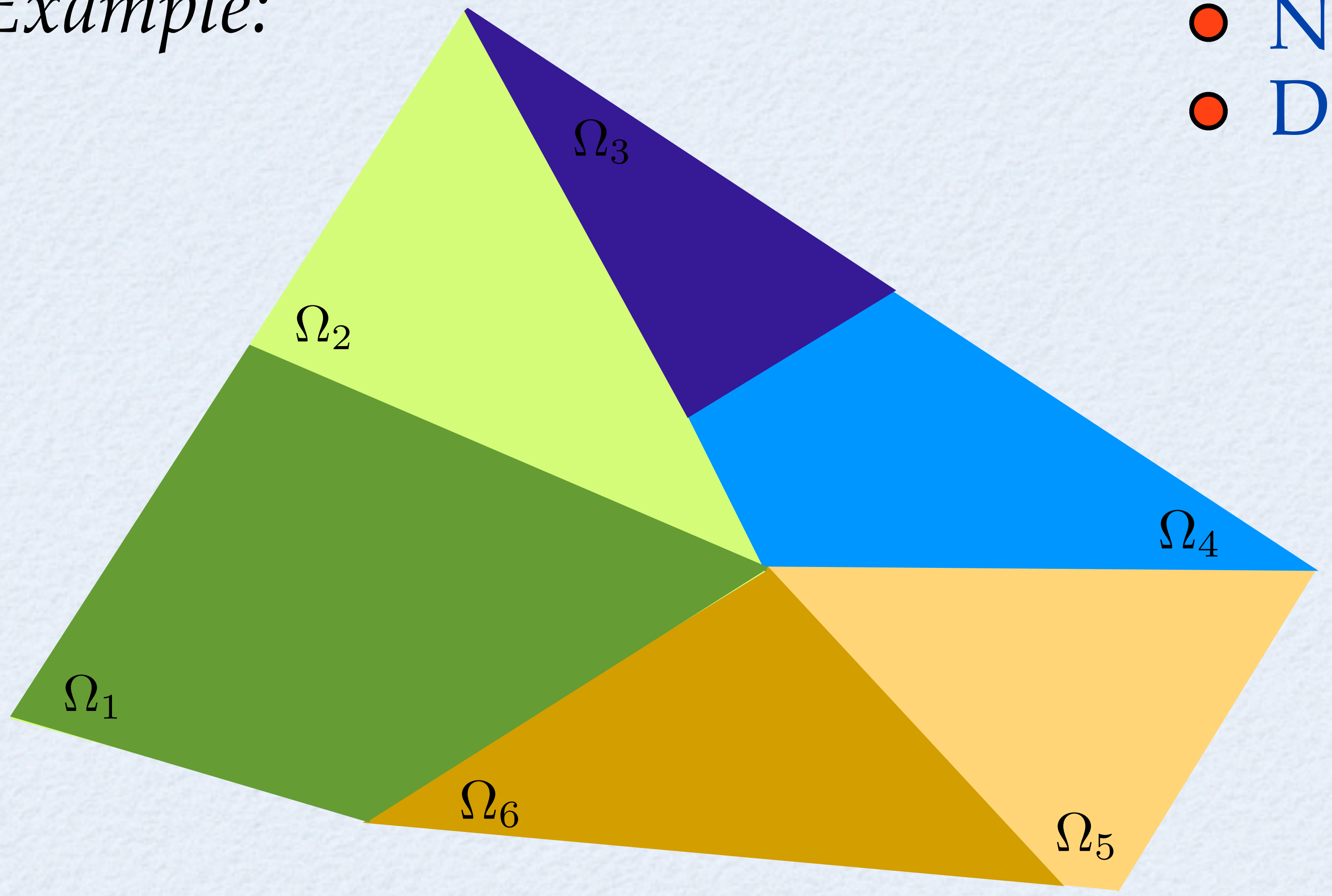
Possible DMS: $3^4 = 81$

Estimation of stochastic hybrid systems

General Estimation Problem

Find from all possible DMS $\mathbf{j}_T$:

Example:



- Number of modes: 6
- Data sequence dimension: 4

$\Downarrow$

Possible DMS: 1296

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Estimation of stochastic hybrid systems

General Estimation Problem

Find from all possible DMS $\mathbf{j}_T$:

- **Estimations**: $\hat{x}_{\mathbf{j}_T}(k|t)\ ,\ \hat{W}_{\mathbf{j}_T}(k|t)\ ,\ \hat{V}_{\mathbf{j}_T}(k|t)$
- Such that:

$$\left\|\hat{Y}^*_{\mathbf{j}_T}(k|t) - Y_T(k)\right\|^2_{\mathbf{\Sigma}^{-1}_{Y_{\mathbf{j}_T}}} \text{ is } \textbf{minimized}$$

- Subject to the following **constraints**:

Dynamic model: $\hat{Y}^*_{\mathbf{j}_T}(k|t) = \mathbf{C}_{\mathbf{j}_T}(k) + \mathbf{D}_{\mathbf{j}_T} U_T(k) + \mathbf{g}_{\mathbf{j}_T} + \mathbf{L}_{\mathbf{j}_T} \hat{W}_{\mathbf{j}_T}(k) + \hat{V}_{\mathbf{j}_T}(k)$

Region bounds: $\mathbf{S}_{\mathbf{j}_T}(k) + \mathbf{R}_{\mathbf{j}_T} U_T(k) + \mathbf{Q}_{\mathbf{j}_T} \hat{W}_{\mathbf{j}_T} \leq \mathbf{T}_{\mathbf{j}_T}$

Disturbance bounds: $\mathbf{H}_{\mathbb{W}_{\mathbf{j}_T}} \hat{W}_{\mathbf{j}_T}(k) \leq \mathbf{h}_{\mathbf{j}_T}\ ,\ \mathbf{H}_{\mathbb{V}_{\mathbf{j}_T}} \hat{V}_{\mathbf{j}_T}(k) \leq \mathbf{h}_{\mathbf{j}_T}$

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Estimation of stochastic hybrid systems

General Estimation Problem

Solution of a **Constrained Least Squares Optimization**:

$$\begin{bmatrix} \hat{x}_{\mathbf{j}_T}(k|t) \\ \hat{W}_{\mathbf{j}_T}(k|t) \\ \hat{V}_{\mathbf{j}_T}(k|t) \end{bmatrix} = \begin{bmatrix} \hat{x}_{\mathbf{j}_T}(k|t-1) \\ \hat{W}_{\mathbf{j}_T}(k|t-1) \\ \hat{V}_{\mathbf{j}_T}(k|t-1) \end{bmatrix} + \mathbf{K}_{\mathbf{j}_T}(k|t) \left(\begin{bmatrix} \mathbf{h_e} \\ \mathbf{h_i} \end{bmatrix} - \begin{bmatrix} \mathbf{H_e} \\ \mathbf{H_i} \end{bmatrix} . \begin{bmatrix} \hat{x}_{\mathbf{j}_T}(k|t-1) \\ \hat{W}_{\mathbf{j}_T}(k|t-1) \\ \hat{V}_{\mathbf{j}_T}(k|t-1) \end{bmatrix} \right)$$

Previous solution
at time instant $t-1$

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Estimation of stochastic hybrid systems

General Estimation Problem

Solution of a **Constrained Least Squares Optimization**:

$$\begin{bmatrix} \hat{x}_{\mathbf{j}_T}(k|t) \\ \hat{W}_{\mathbf{j}_T}(k|t) \\ \hat{V}_{\mathbf{j}_T}(k|t) \end{bmatrix} = \begin{bmatrix} \hat{x}_{\mathbf{j}_T}(k|t-1) \\ \hat{W}_{\mathbf{j}_T}(k|t-1) \\ \hat{V}_{\mathbf{j}_T}(k|t-1) \end{bmatrix} + \mathbf{K}_{\mathbf{j}_T}(k|t) \left(\begin{bmatrix} \mathbf{h_e} \\ \mathbf{h_i} \end{bmatrix} - \begin{bmatrix} \mathbf{H_e} \\ \mathbf{H_i} \end{bmatrix} \cdot \begin{bmatrix} \hat{x}_{\mathbf{j}_T}(k|t-1) \\ \hat{W}_{\mathbf{j}_T}(k|t-1) \\ \hat{V}_{\mathbf{j}_T}(k|t-1) \end{bmatrix} \right)$$

Dynamic model

$$\begin{bmatrix} \hat{Y}^*_{\mathbf{j}_T}(k|t) - \mathbf{D}_{\mathbf{j}_T} U_T(k) - \mathbf{g}_{\mathbf{j}_T} \\ \mathbf{T}_{\mathbf{j}_T} - \mathbf{R}_{\mathbf{j}_T} U_T(k) \\ \mathbf{h}_{\mathbb{W}_{\mathbf{j}_T}} \\ \mathbf{h}_{\mathbb{V}_{\mathbf{j}_T}} \end{bmatrix} - \begin{bmatrix} \mathbf{C}_{\mathbf{j}_T} & \mathbf{L}_{\mathbf{j}_T} & I_{n_Y} \\ \mathbf{S}_{\mathbf{j}_T} & \mathbf{Q}_{\mathbf{j}_T} & 0 \\ 0 & \mathbf{H}_{\mathbb{W}_{\mathbf{j}_T}} & 0 \\ 0 & 0 & \mathbf{H}_{\mathbb{V}_{\mathbf{j}_T}} \end{bmatrix} \cdot \begin{bmatrix} \hat{x}_{\mathbf{j}_T}(k|t-1) \\ \hat{W}_{\mathbf{j}_T}(k|t-1) \\ \hat{V}_{\mathbf{j}_T}(k|t-1) \end{bmatrix}$$

Region bounds

Disturbance bounds

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Estimation of stochastic hybrid systems

General Estimation Problem

Solution of a **Constrained Least Squares Optimization**:

$$\begin{bmatrix} \hat{x}_{\mathbf{j}_T}(k|t) \\ \hat{W}_{\mathbf{j}_T}(k|t) \\ \hat{V}_{\mathbf{j}_T}(k|t) \end{bmatrix} = \begin{bmatrix} \hat{x}_{\mathbf{j}_T}(k|t-1) \\ \hat{W}_{\mathbf{j}_T}(k|t-1) \\ \hat{V}_{\mathbf{j}_T}(k|t-1) \end{bmatrix} + \mathbf{K}_{\mathbf{j}_T}(k|t) \left(\begin{bmatrix} \mathbf{h_e} \\ \mathbf{h_i} \end{bmatrix} - \begin{bmatrix} \mathbf{H_e} \\ \mathbf{H_i} \end{bmatrix} \cdot \begin{bmatrix} \hat{x}_{\mathbf{j}_T}(k|t-1) \\ \hat{W}_{\mathbf{j}_T}(k|t-1) \\ \hat{V}_{\mathbf{j}_T}(k|t-1) \end{bmatrix} \right)$$

$$\mathbf{K}_{\mathbf{j}_T}(k|t) = \left(\begin{bmatrix} \mathbf{\Sigma}_{x_{\mathbf{j}_T}}(k|t-1) & 0 & 0 \\ 0 & \mathbf{\Sigma}_{W_{\mathbf{j}_T}} & 0 \\ 0 & 0 & \mathbf{\Sigma}_{V_{\mathbf{j}_T}} \end{bmatrix}^{-1} + \begin{bmatrix} \mathbf{H_e} \\ \mathbf{H_i} \end{bmatrix}^{\mathrm{T}} \mathbf{Z}_{\mathbf{j}_T}(k|t) \begin{bmatrix} \mathbf{H_e} \\ \mathbf{H_i} \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathbf{H_e} \\ \mathbf{H_i} \end{bmatrix}^{\mathrm{T}} \mathbf{Z}_{\mathbf{j}_T}(k|t)$$

Covariance matrix

Active set constraints matrix

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Estimation of stochastic hybrid systems

General Estimation Problem

Solution of a **Constrained Least Squares Optimization**:

$$\begin{bmatrix} \hat{x}_{\mathbf{j}_T}(k|t) \\ \hat{W}_{\mathbf{j}_T}(k|t) \\ \hat{V}_{\mathbf{j}_T}(k|t) \end{bmatrix} = \begin{bmatrix} \hat{x}_{\mathbf{j}_T}(k|t-1) \\ \hat{W}_{\mathbf{j}_T}(k|t-1) \\ \hat{V}_{\mathbf{j}_T}(k|t-1) \end{bmatrix} + \mathbf{K}_{\mathbf{j}_T}(k|t) \left(\begin{bmatrix} \mathbf{h_e} \\ \mathbf{h_i} \end{bmatrix} - \begin{bmatrix} \mathbf{H_e} \\ \mathbf{H_i} \end{bmatrix} \cdot \begin{bmatrix} \hat{x}_{\mathbf{j}_T}(k|t-1) \\ \hat{W}_{\mathbf{j}_T}(k|t-1) \\ \hat{V}_{\mathbf{j}_T}(k|t-1) \end{bmatrix} \right)$$

Complex and time-consuming optimization!

The Interacting Multiple Model

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Estimation of stochastic hybrid systems

The Interacting Multiple Model

- **1st step:** unconstrained least squares optimization

For all possible DMS: $\mathbf{j}_T$

- **Find**: $\hat{x}_{\mathbf{j}_T}(k|t)$, ~~$\hat{W}_{\mathbf{j}_T}(k|t)$~~ , ~~$\hat{V}_{\mathbf{j}_T}(k|t)$~~
- Such that:

$$\left\| \hat{Y}^*_{\mathbf{j}_T}(k|t) - Y_T(k) \right\|^2_{\mathbf{\Sigma}^{-1}_{Y_{\mathbf{j}_T}}} \text{ is } \textbf{minimized}$$

- Under to the following **constraints**:

Dynamic model: $\hat{Y}^*_{\mathbf{j}_T}(k|t) = \mathbf{C}_{\mathbf{j}_T}(k) + \mathbf{D}_{\mathbf{j}_T} U_T(k) + \mathbf{g}_{\mathbf{j}_T}$ ~~$+ \mathbf{L}_{\mathbf{j}_T} \hat{W}_{\mathbf{j}_T}(k) + \hat{V}_{\mathbf{j}_T}(k)$~~

Region bounds: ~~$\mathbf{S}_{\mathbf{j}_T}(k) + \mathbf{R}_{\mathbf{j}_T} U_T(k) + \mathbf{Q}_{\mathbf{j}_T} \hat{W}_{\mathbf{j}_T} \leq \mathbf{T}_{\mathbf{j}_T}$~~

Disturbance bounds: ~~$\mathbf{H}_{\mathbb{W}_{\mathbf{j}_T}} \hat{W}_{\mathbf{j}_T}(k) \leq \mathbf{h}_{\mathbf{j}_T}$, $\mathbf{H}_{\mathbb{V}_{\mathbf{j}_T}} \hat{V}_{\mathbf{j}_T}(k) \leq \mathbf{h}_{\mathbf{j}_T}$~~

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Estimation of stochastic hybrid systems

The Interacting Multiple Model

- **1st step:** **unconstrained least squares optimization**

$$\hat{x}_{\mathbf{j}_T}(k|t) = \hat{x}_{\mathbf{j}_T}(k|t-1) + \mathbf{K}_{\mathbf{j}_T}(k|t-1)\left[Y_T(k) - \hat{Y}^*_{\mathbf{j}_T}(k|t)\right]$$

Advantage: fast optimization.

Concern: some DMS $\mathbf{j}_T$ admit *unfeasible state trajectories* $\hat{x}_{\mathbf{j}_T}(k)$ if region bounds were to be considered.

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Estimation of stochastic hybrid systems

- **2nd step:** ranks the DMS according to the optimization error

$$\left\| \hat{Y}^*_{\mathbf{j}_T}(k|t) - Y_T(k) \right\|^2 = \epsilon^u_{\mathbf{j}_T}$$

DMS	Error
$\mathbf{j}_T^1$	$\epsilon^u_{\mathbf{j}_T^1}$
$\mathbf{j}_T^2$	$\epsilon^u_{\mathbf{j}_T^2}$
$\mathbf{j}_T^3$	$\epsilon^u_{\mathbf{j}_T^3}$
$\mathbf{j}_T^4$	$\epsilon^u_{\mathbf{j}_T^4}$
$\vdots$	$\vdots$
$\mathbf{j}_T^n$	$\epsilon^u_{\mathbf{j}_T^n}$

Ascending order:

$$\epsilon^u_{\mathbf{j}_T^1} < \epsilon^u_{\mathbf{j}_T^2} < \cdots < \epsilon^u_{\mathbf{j}_T^n}$$

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Estimation of stochastic hybrid systems

- **3rd step: constrained least squares optimization**

DMS	Error
$\mathbf{j}_T^1$	$\epsilon^u_{\mathbf{j}_T^1}$
$\mathbf{j}_T^2$	$\epsilon^u_{\mathbf{j}_T^2}$
$\mathbf{j}_T^3$	$\epsilon^u_{\mathbf{j}_T^3}$
$\mathbf{j}_T^4$	$\epsilon^u_{\mathbf{j}_T^4}$
$\vdots$	$\vdots$
$\mathbf{j}_T^n$	$\epsilon^u_{\mathbf{j}_T^n}$

$\mathbf{j}_T^1$ → Solve constrained least squares

For the DMS: $\mathbf{j}_T^1$

- **Find**: $\hat{x}_{\mathbf{j}_T^1}(k|t)$, $\hat{W}_{\mathbf{j}_T^1}(k|t)$, $\hat{V}_{\mathbf{j}_T^1}(k|t)$
- Such that:

$$\left\| \hat{Y}^*_{\mathbf{j}_T^1}(k|t) - Y_T(k) \right\|^2 \text{ is } \textbf{minimized}$$

- Subjecto to the following **constraints**:

Dynamic model: $\hat{Y}^*_{\mathbf{j}_T^1}(k|t) = \mathbf{C}_{\mathbf{j}_T^1}(k) + \mathbf{D}_{\mathbf{j}_T^1} U_T(k) + \mathbf{g}_{\mathbf{j}_T^1} + \mathbf{L}_{\mathbf{j}_T^1} \hat{W}_{\mathbf{j}_T^1}(k) + \hat{V}_{\mathbf{j}_T^1}(k)$

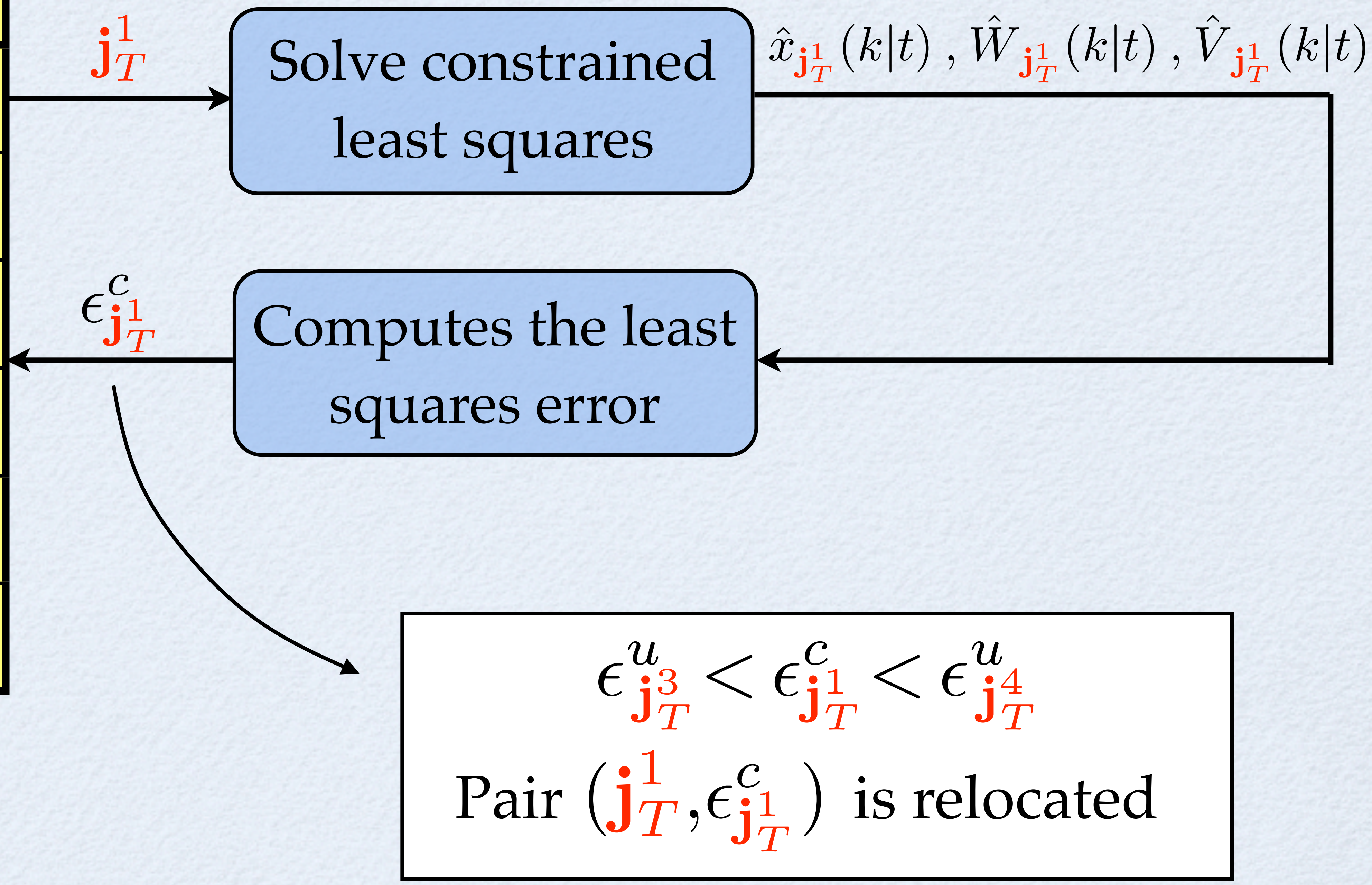
Region bounds: $\mathbf{S}_{\mathbf{j}_T^1}(k) + \mathbf{R}_{\mathbf{j}_T^1} U_T(k) + \mathbf{Q}_{\mathbf{j}_T^1} \hat{W}_{\mathbf{j}_T^1} \leq \mathbf{T}_{\mathbf{j}_T^1}$

Disturbance bounds: $\mathbf{H}_{\mathbb{W}_{\mathbf{j}_T^1}} \hat{W}_{\mathbf{j}_T^1}(k) \leq \mathbf{h}_{\mathbf{j}_T^1}$, $\mathbf{H}_{\mathbb{V}_{\mathbf{j}_T^1}} \hat{V}_{\mathbf{j}_T^1}(k) \leq \mathbf{h}_{\mathbf{j}_T^1}$

Estimation of stochastic hybrid systems

- **3rd step: constrained least squares optimization**

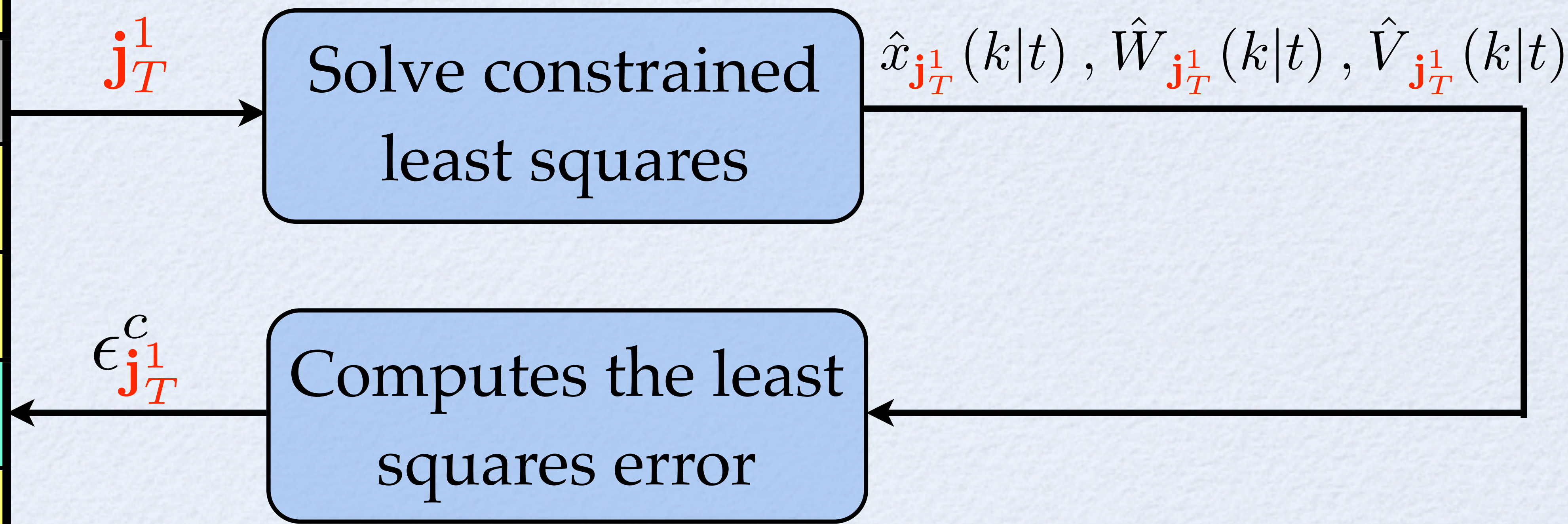
DMS	Error
$\mathbf{j}_T^1$	$\epsilon^u_{\mathbf{j}_T^1}$
$\mathbf{j}_T^2$	$\epsilon^u_{\mathbf{j}_T^2}$
$\mathbf{j}_T^3$	$\epsilon^u_{\mathbf{j}_T^3}$
$\mathbf{j}_T^4$	$\epsilon^u_{\mathbf{j}_T^4}$
$\vdots$	$\vdots$
$\mathbf{j}_T^n$	$\epsilon^u_{\mathbf{j}_T^n}$



Estimation of stochastic hybrid systems

- **3rd step:** **constrained least squares optimization**

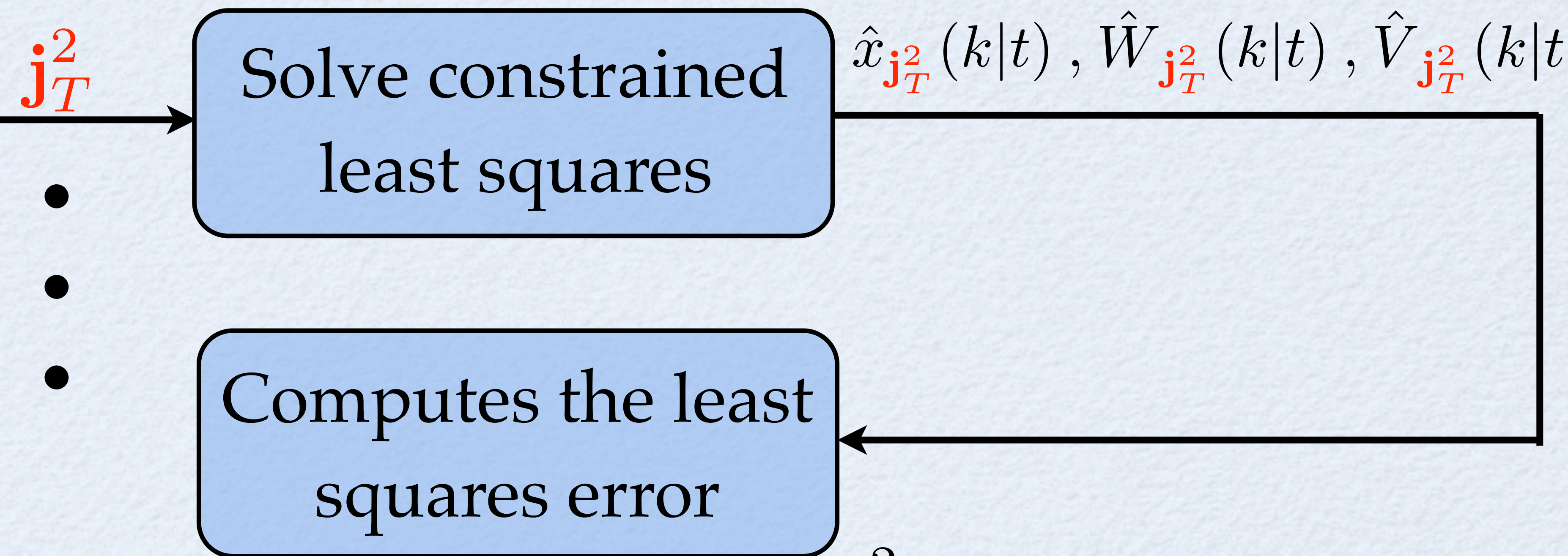
DMS	Error
$\mathbf{j}_T^1$	$\epsilon^u_{\mathbf{j}_T^1}$
$\mathbf{j}_T^2$	$\epsilon^u_{\mathbf{j}_T^2}$
$\mathbf{j}_T^3$	$\epsilon^u_{\mathbf{j}_T^3}$
$\mathbf{j}_T^1$	$\epsilon^c_{\mathbf{j}_T^1}$
$\mathbf{j}_T^4$	$\epsilon^u_{\mathbf{j}_T^4}$
$\vdots$	$\vdots$
$\mathbf{j}_T^n$	$\epsilon^u_{\mathbf{j}_T^n}$



Estimation of stochastic hybrid systems

- **3rd step: constrained least squares optimization**

DMS	Error
$\mathbf{j}_T^1$	$\epsilon^u_{\mathbf{j}_T^1}$
$\mathbf{j}_T^2$	$\epsilon^u_{\mathbf{j}_T^2}$
$\mathbf{j}_T^3$	$\epsilon^u_{\mathbf{j}_T^3}$
$\mathbf{j}_T^1$	$\epsilon^c_{\mathbf{j}_T^1}$
$\mathbf{j}_T^4$	$\epsilon^u_{\mathbf{j}_T^4}$
$\vdots$	$\vdots$
$\mathbf{j}_T^n$	$\epsilon^u_{\mathbf{j}_T^n}$



$$\epsilon^c_{\mathbf{j}_T^2} = \left\| \hat{Y}^*_{\mathbf{j}_T^2}(k|t) - Y_T(k) \right\|^2$$

Estimation of stochastic hybrid systems

- **3rd step:** **constrained least squares optimization**

DMS	Error
$\mathbf{j}_T^1$	$\epsilon^u_{\mathbf{j}_T^1}$
$\mathbf{j}_T^2$	$\epsilon^u_{\mathbf{j}_T^2}$
$\mathbf{j}_T^3$	$\epsilon^u_{\mathbf{j}_T^3}$
$\mathbf{j}_T^1$	$\epsilon^c_{\mathbf{j}_T^1}$
$\mathbf{j}_T^4$	$\epsilon^u_{\mathbf{j}_T^4}$
$\vdots$	$\vdots$
$\mathbf{j}_T^n$	$\epsilon^u_{\mathbf{j}_T^n}$

$\mathbf{j}_T^2$ → Solve constrained least squares → $\hat{x}_{\mathbf{j}_T^2}(k|t)\ ,\ \hat{W}_{\mathbf{j}_T^2}(k|t)\ ,\ \hat{V}_{\mathbf{j}_T^2}(k|t)$ → Computes the least squares error

$$\epsilon^c_{\mathbf{j}_T^2} = \left\| \hat{Y}^*_{\mathbf{j}_T^2}(k|t) - Y_T(k) \right\|^2$$

Process stops when $\mathbf{j}_T^i$ is already a constrained solution $\Longrightarrow$ Optimal solution found!

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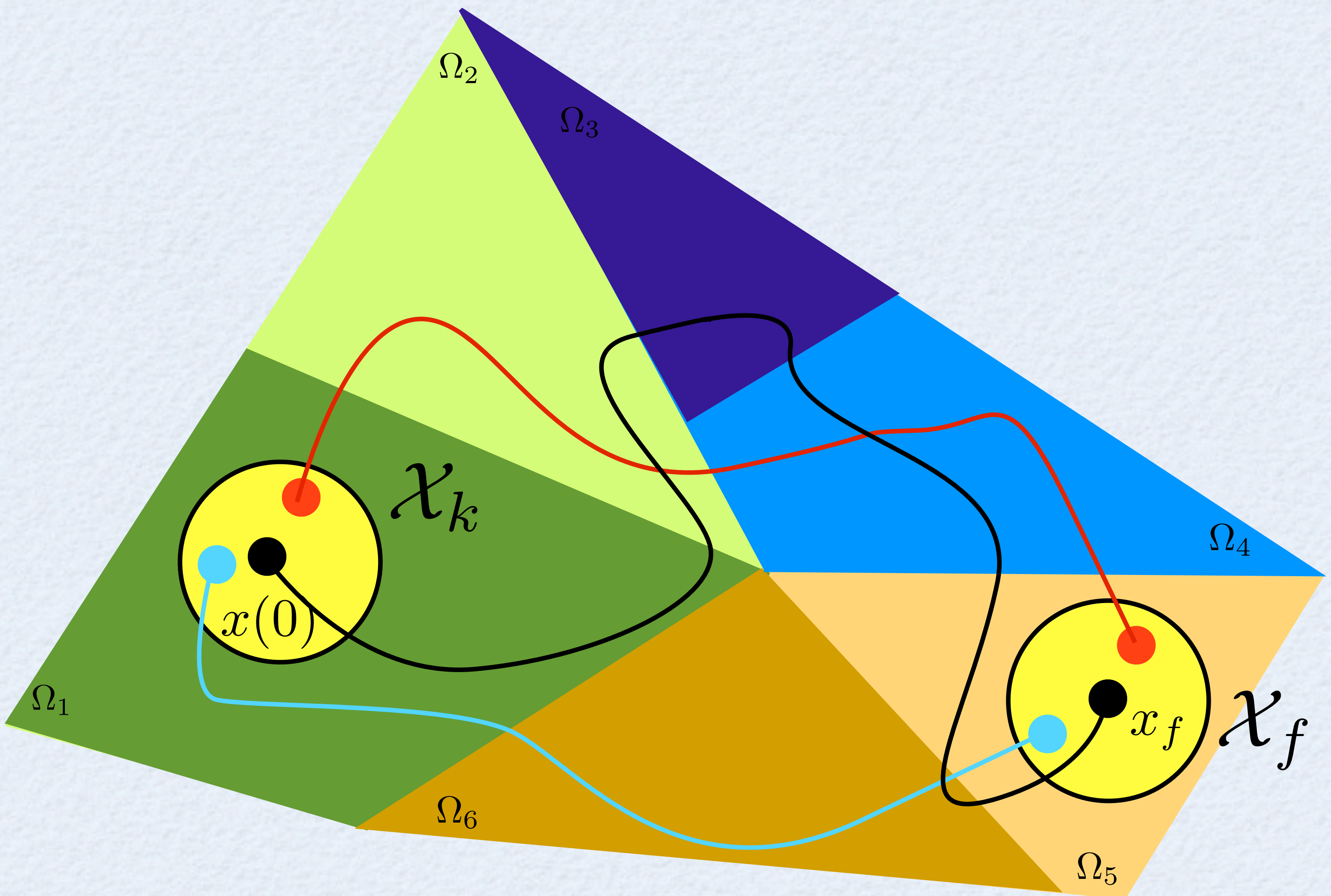
Part III

OPTIMAL CONTROL OF STOCHASTIC HYBRID SYSTEMS

Optimal control of stochastic hybrid systems

Problem formulation



- Move in N steps from:

$$x(0) \longrightarrow x_f$$

- Uncertainty in the state:

$$w(k) \in \mathbb{W}$$
$$x(k) \in \mathcal{X}_k$$

- Relaxation:

$$\mathcal{X}_k \rightarrow \mathcal{X}_f$$

Find: $\left\{\begin{array}{l}\text{the optimal DMS of length N: } \mathbf{i}_N^* \\ \text{the N optimal control moves: } u_N^*\end{array}\right.$ and apply **RHC**.

Optimal control of stochastic hybrid systems

- **Find:** $\{\mathbf{i}_N^*, u_N^*\}$

- **Resulting from:**

$$\arg\left\{\min_{\substack{\mathbf{i}_N \in \mathcal{I} \\ u_N \in \mathcal{U}}} \max_{\substack{w(k)\in\mathbb{W} \\ x(k)\in\mathcal{X}_k}} \left(\sum_{t=k}^{k+N-1} \min_{a(t|k)\in\mathcal{X}_f} \|x(t|k)-a(t|k)\| + \|u(t|k)-u_f\| \right)\right\}$$

For the worst case scenario of disturbances affecting $x(k)$

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Optimal control of stochastic hybrid systems

- **Find:** $\{\mathbf{i}_N^*, u_N^*\}$

- **Resulting from:**

$$\arg\left\{ \min_{\substack{\mathbf{i}_N\in\mathcal{I}\\ u_N\in\mathcal{U}}} \max_{\substack{w(k)\in\mathbb{W}\\ x(k)\in\mathcal{X}_k}} \left(\sum_{t=k}^{k+N-1} \min_{a(t|k)\in\mathcal{X}_f} \|x(t|k)-a(t|k)\| + \|u(t|k)-u_f\| \right) \right\}$$

Minimize with respect to:

$$i(k)\ldots i(k+N-1)$$

and

$$u(k)\ldots u(k+N-1)$$

Optimal control of stochastic hybrid systems

- **Find:** $\{\mathbf{i}_N^*, u_N^*\}$

- **Resulting from:**

$$\arg\left\{\min_{\substack{\mathbf{i}_N\in\mathcal{I}\\ u_N\in\mathcal{U}}}\max_{\substack{w(k)\in\mathbb{W}\\ x(k)\in\mathcal{X}_k}}\left(\sum_{t=k}^{k+N-1}\min_{a(t|k)\in\mathcal{X}_f}\|x(t|k)-a(t|k)\|+\|u(t|k)-u_f\|\right)\right\}$$

Minimum distance between the state and the auxiliary variable $a(t|k)\in\mathcal{X}_f$

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Optimal control of stochastic hybrid systems

- **Find:** $\{\mathbf{i}_N^*, u_N^*\}$

- **Resulting from:**

$$\arg\left\{\min_{\substack{\mathbf{i}_N\in\mathcal{I}\\ u_N\in\mathcal{U}}}\max_{\substack{w(k)\in\mathbb{W}\\ x(k)\in\mathcal{X}_k}}\left(\sum_{t=k}^{k+N-1}\min_{a(t|k)\in\mathcal{X}_f}\|x(t|k)-a(t|k)\|+\|u(t|k)-u_f\|\right)\right\}$$

The state reaches steady state nominal input: $u(k|t)=u_f$

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Optimal control of stochastic hybrid systems

- **Find:** $\{\mathbf{i}_N^*, u_N^*\}$

- **Resulting from:**

$$\arg\left\{\min_{\substack{\mathbf{i}_N\in\mathcal{I}\\ u_N\in\mathcal{U}}}\ \max_{\substack{w(k)\in\mathbb{W}\\ x(k)\in\mathcal{X}_k}}\left(\sum_{t=k}^{k+N-1}\min_{a(t|k)\in\mathcal{X}_f}\|x(t|k)-a(t|k)\|+\|u(t|k)-u_f\|\right)\right\}$$

- **Subject to the following constraints:**

Dynamic model: $x(k+1)=A_{i(k)}\,x(k)+B_{i(k)}\,u(k)+W_{i(k)}w(k)+f_{i(k)}$

Region bounds: $\Omega_i \triangleq \left\{\begin{bmatrix} x(k)\\ u(k)\\ w(k)\end{bmatrix} : S_i x(k)+R_i u(k)+Q_i w(k)\leq T_i\right\}$

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Optimal control of stochastic hybrid systems

- **Find:** $\{\mathbf{i}_N^*, u_N^*\}$

- **Resulting from:**

$$\arg\left\{\min_{\substack{\mathbf{i}_N\in\mathcal{I}\\ u_N\in\mathcal{U}}}\ \max_{\substack{w(k)\in\mathbb{W}\\ x(k)\in\mathcal{X}_k}}\left(\sum_{t=k}^{k+N-1}\min_{a(t|k)\in\mathcal{X}_f}\|x(t|k)-a(t|k)\|+\|u(t|k)-u_f\|\right)\right\}$$

INFINITE DIMENSION

NON-CONVEX

MIXED-INTEGER OPTIMIZATION

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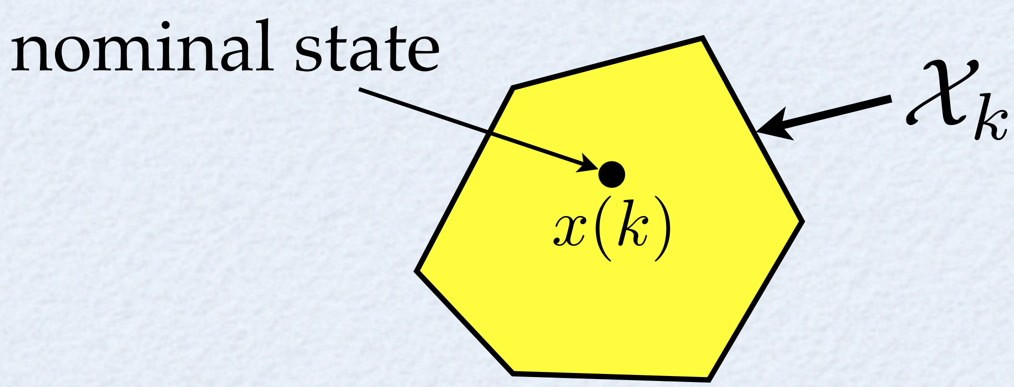
Optimal control of stochastic hybrid systems

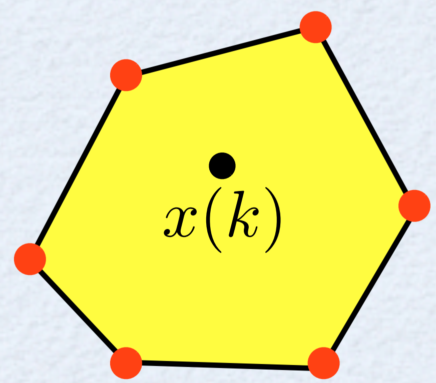
Robust Mode Control

- Disturbances $\mathbb{W}$ are always a **bounded convex polytope**:



- The maximum of a convex function over a convex set $\mathcal{X}_k$ is found at one of their vertices:

$\bullet \equiv \Upsilon_{\mathcal{X}}$



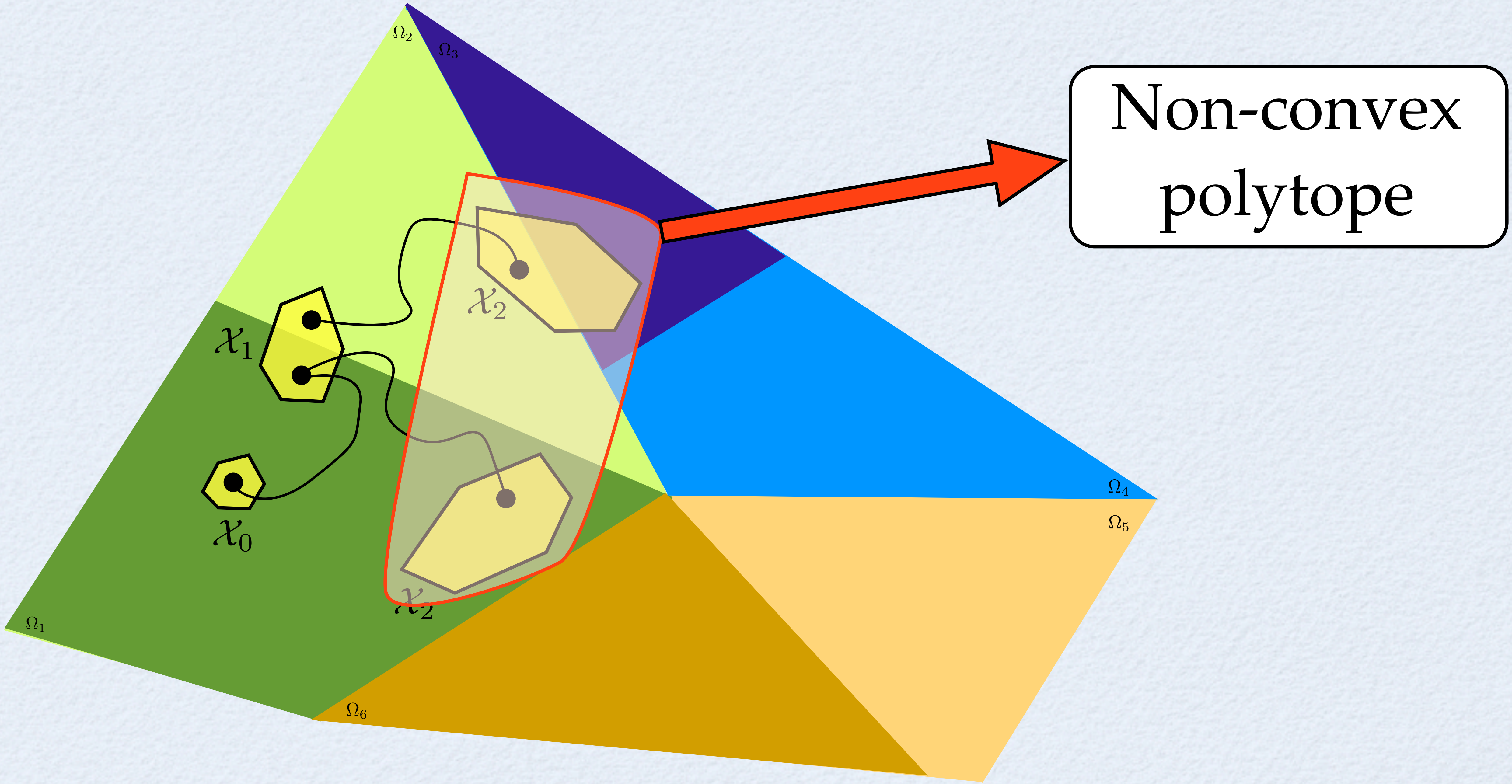
Finite dimension optimization

$$\max_{\substack{w(k)\in\mathbb{W}\\ x(k)\in\mathcal{X}_k}} \equiv \max_{\substack{w(k)\in\Upsilon_{\mathbb{W}}\\ x(k)\in\Upsilon_{\mathcal{X}}}}$$

Optimal control of stochastic hybrid systems

Robust Mode Control

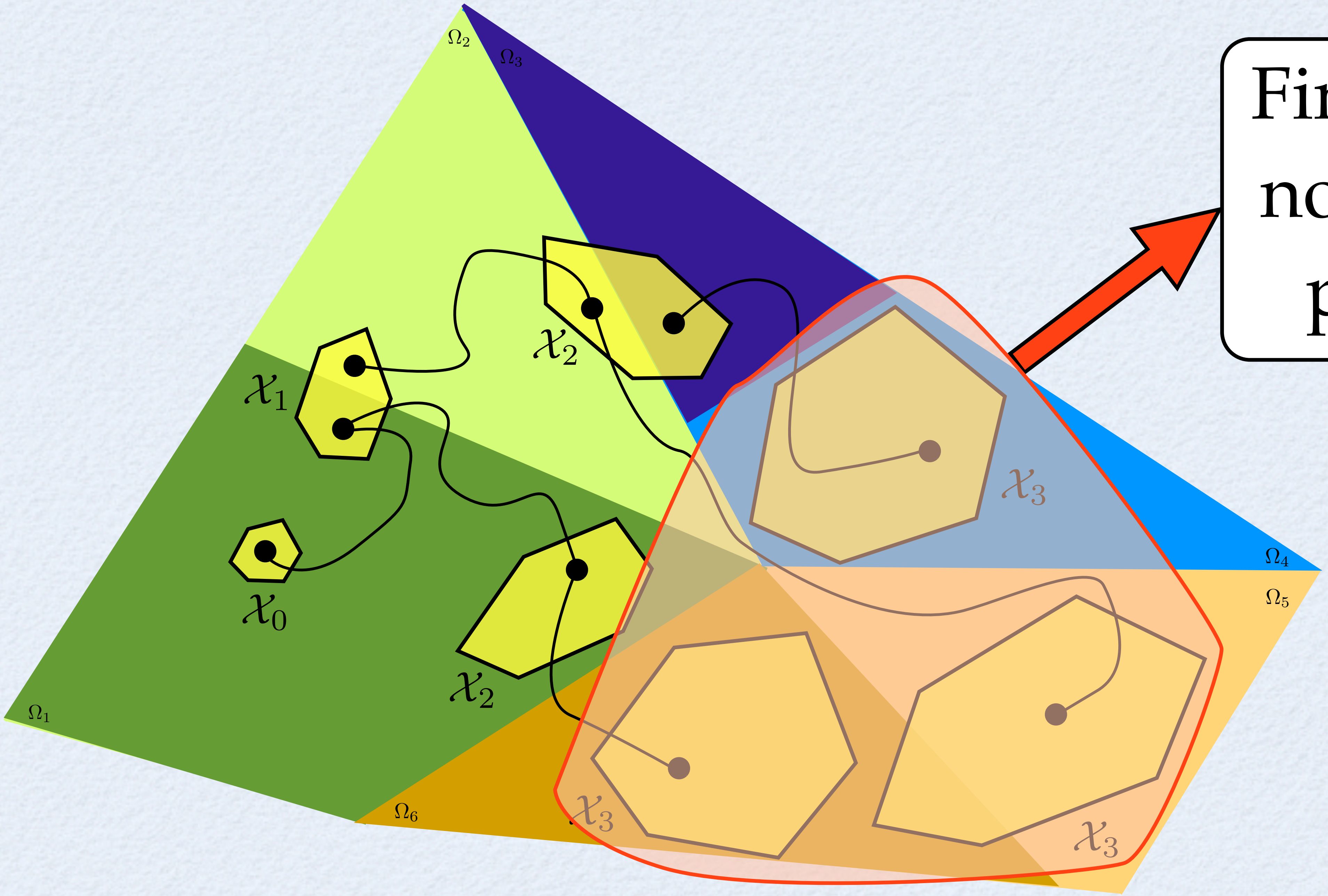
- The final disturbed state prediction polytope is non-convex:



Optimal control of stochastic hybrid systems

Robust Mode Control

- The final disturbed state prediction polytope is non-convex:

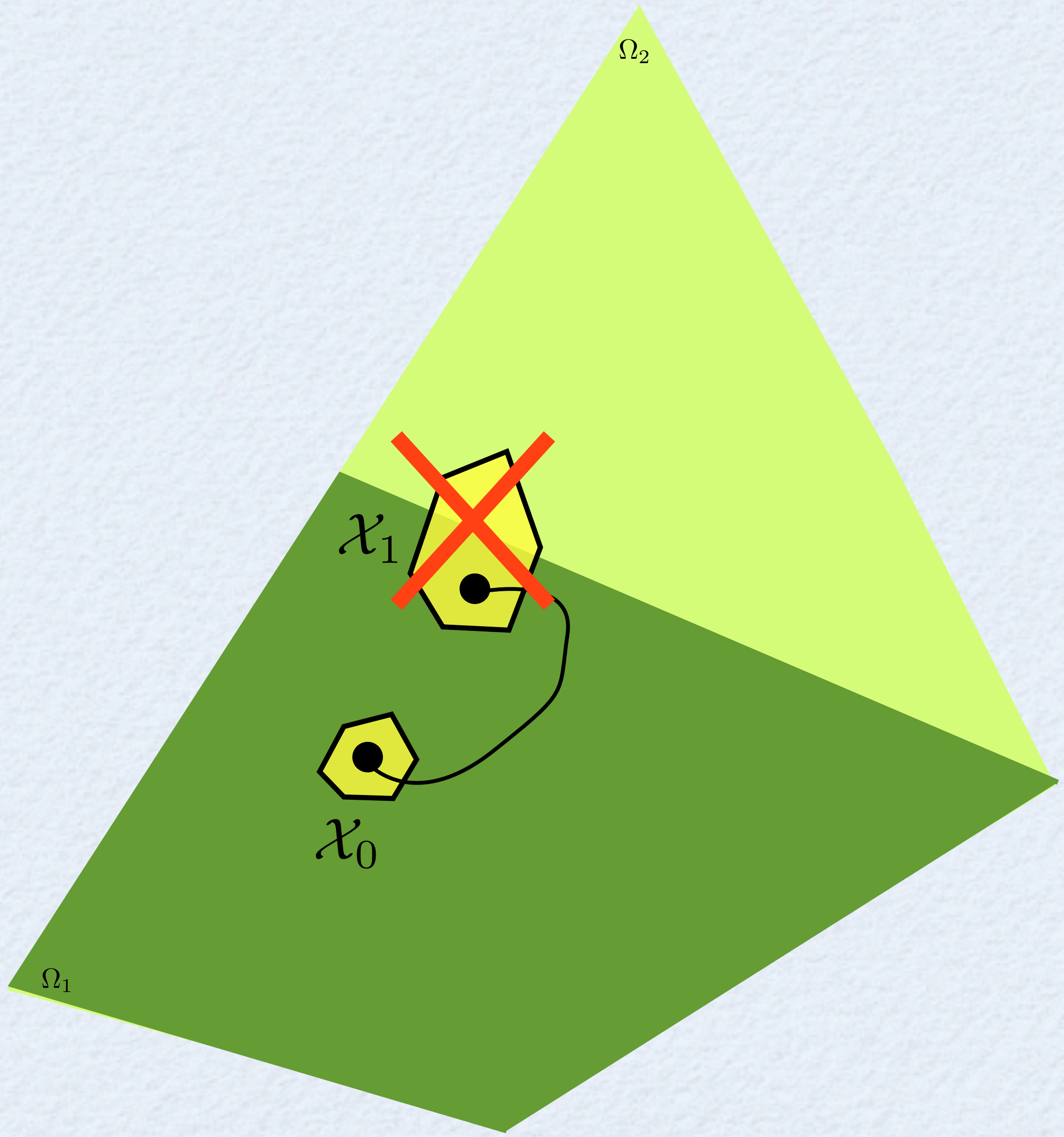


Final set is a non-convex polytope

Optimal control of stochastic hybrid systems

Robust Mode Control

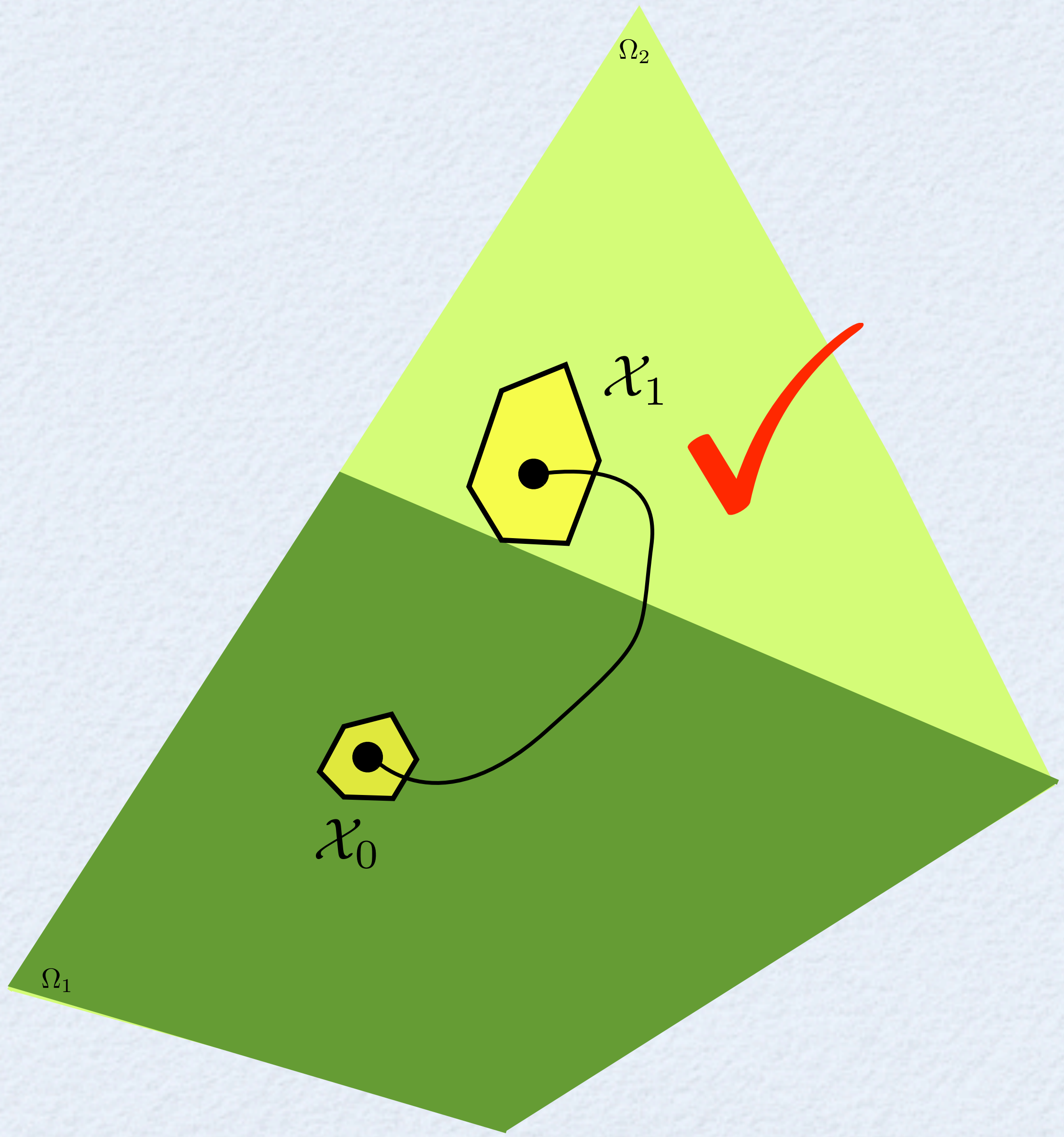
- **Key idea:** restrict the control moves such that, for every value of the disturbances, the **mode is unique** at each time instant k.



Optimal control of stochastic hybrid systems

Robust Mode Control

- **Key idea:** restrict the control moves such that, for every value of the disturbances, the **mode is unique** at each time instant k.

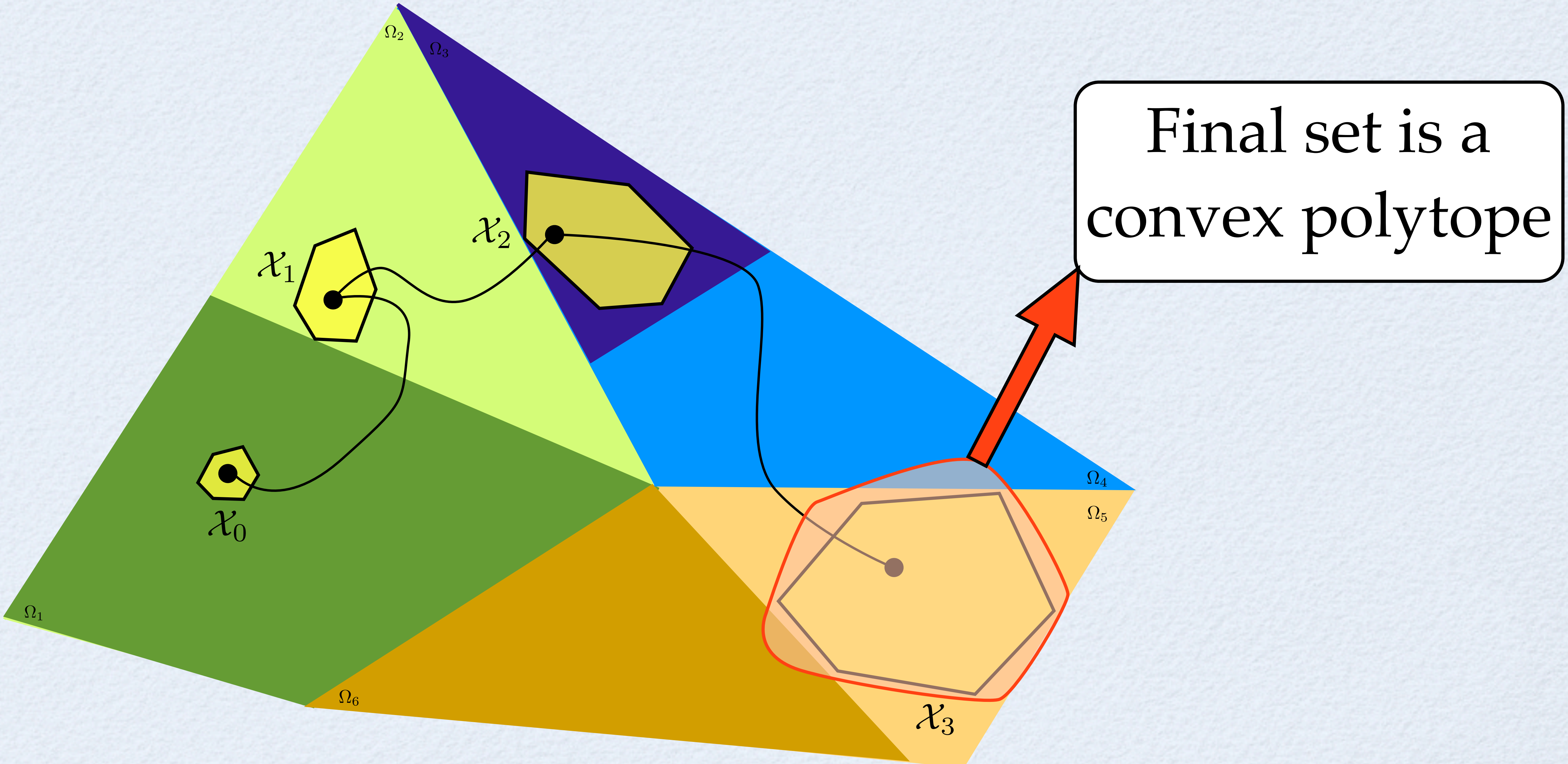


The disturbed set lives only in one mode

Optimal control of stochastic hybrid systems

Robust Mode Control

- **Key idea:** the **mode is unique** at each time instant k.



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Optimal control of stochastic hybrid systems

Robust Mode Control

- For all feasible DMS, $j(k) \ldots j(k+N-1)$, find: u_N^*
- Resulting from:

$$\arg\left\{\min_{\mathbf{u}_N \in \mathcal{U}} \max_{\substack{w(k)\in\Upsilon_{\mathbb{W}} \\ x(k)\in\Upsilon_{\mathcal{X}}}} \left(\sum_{t=k}^{k+N-1} \min_{a(t|k)\in\mathcal{X}_f} \|x(t|k)-a(t|k)\|+\|u(t|k)-u_f\|\right)\right\}$$

- Subject to the following constraints:

Dynamic model: $x(k+1) = A_{j(k)}\,x(k) + B_{j(k)}\,u(k) + W_{j(k)}w(k) + f_{j(k)}$

Robust mode control: $S_{j(k)}x(k) + R_{j(k)}u(k) + Q_{j(k)}w(k) \leq T_{j(k)}$

Bounded convex disturbances: $H_{\mathbb{W}_{j(k)}}w(k) \leq h_{\mathbb{W}_{j(k)}}$

Optimal control of stochastic hybrid systems

Robust Mode Control

- For all feasible DMS, $j(k) \ldots j(k+N-1)$, find: u_N^*
- Resulting from:

$$\arg\left\{ \min_{\mathbf{u}_N \in \mathcal{U}} \max_{\substack{w(k)\in\Upsilon_{\mathbb{W}} \\ x(k)\in\Upsilon_{\mathcal{X}}}} \left(\sum_{t=k}^{k+N-1} \min_{a(t|k)\in\mathcal{X}_f} \|x(t|k)-a(t|k)\| + \|u(t|k)-u_f\| \right) \right\}$$

FINITE DIMENSION

CONVEX

MIXED-INTEGER OPTIMIZATION

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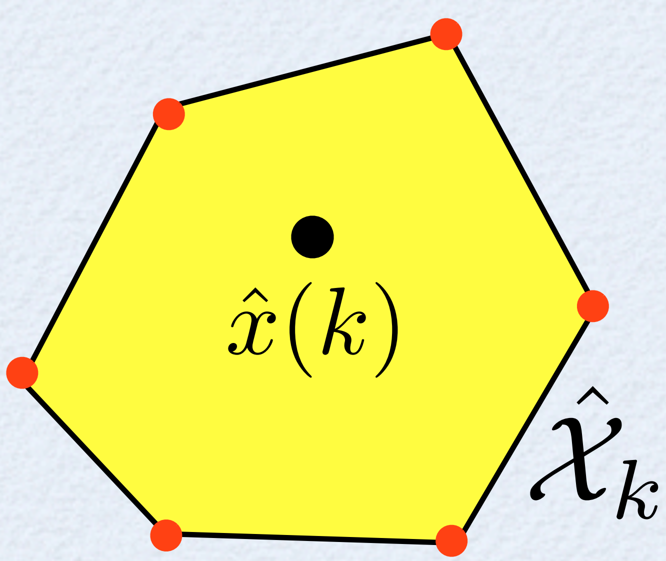
Optimal control of stochastic hybrid systems

Estimation & Robust Mode Control

- How does estimation helps control?

Estimator: $\hat{\mathbf{i}}_T(k)$, $\hat{x}_{\hat{\mathbf{i}}_T}(k)$, $\hat{w}_{\hat{\mathbf{i}}_T}(k)$, $\hat{v}_{\hat{\mathbf{i}}_T}(k)$

At the current time instant k: $\hat{x}(k)$, $\hat{w}(k)$ ✓



If the estimation was perfect $\hat{\mathcal{X}}_k$ would be reduced to $\hat{x}(k)$

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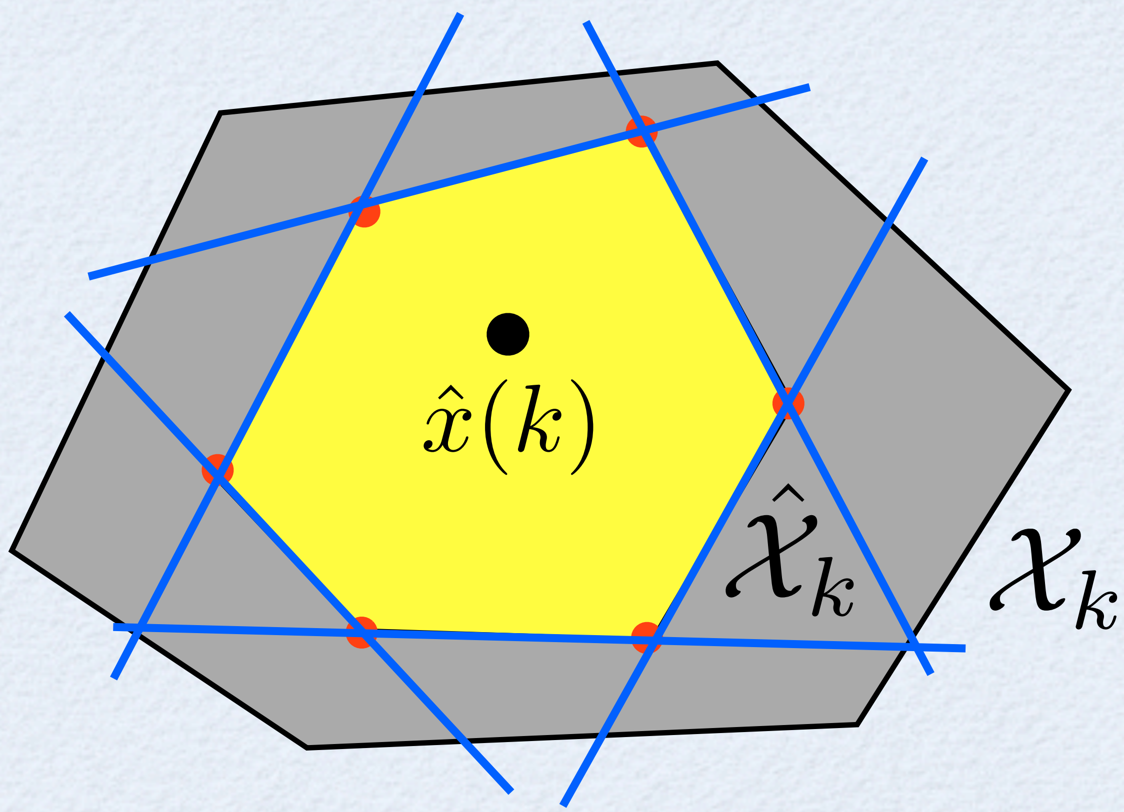
Optimal control of stochastic hybrid systems

Estimation & Robust Mode Control

- How does estimation helps control?

Estimator: $\hat{\mathbf{i}}_T(k)$, $\hat{x}_{\hat{\mathbf{i}}_T}(k)$, $\hat{w}_{\hat{\mathbf{i}}_T}(k)$, $\hat{v}_{\hat{\mathbf{i}}_T}(k)$



- *Reduced uncertainty.*
- *More accurate state predictions over the control horizon.*
- *Improves control performance.*

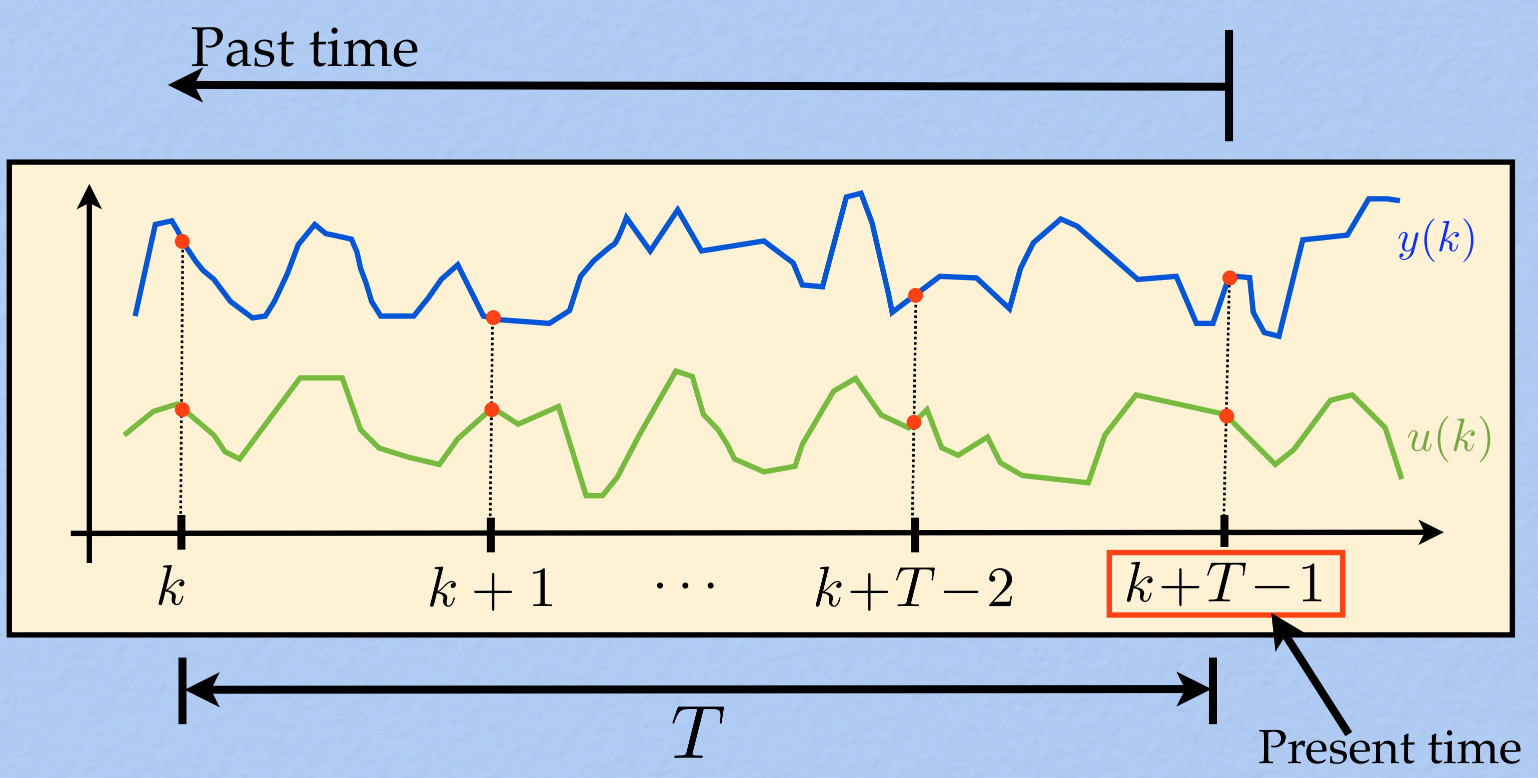
Optimal control of stochastic hybrid systems

Estimation & Robust Mode Control

- How does Robust Mode Control helps estimation?

Looks T instants into the past to estimate the DMS:

$\hat{\mathbf{i}}_T$

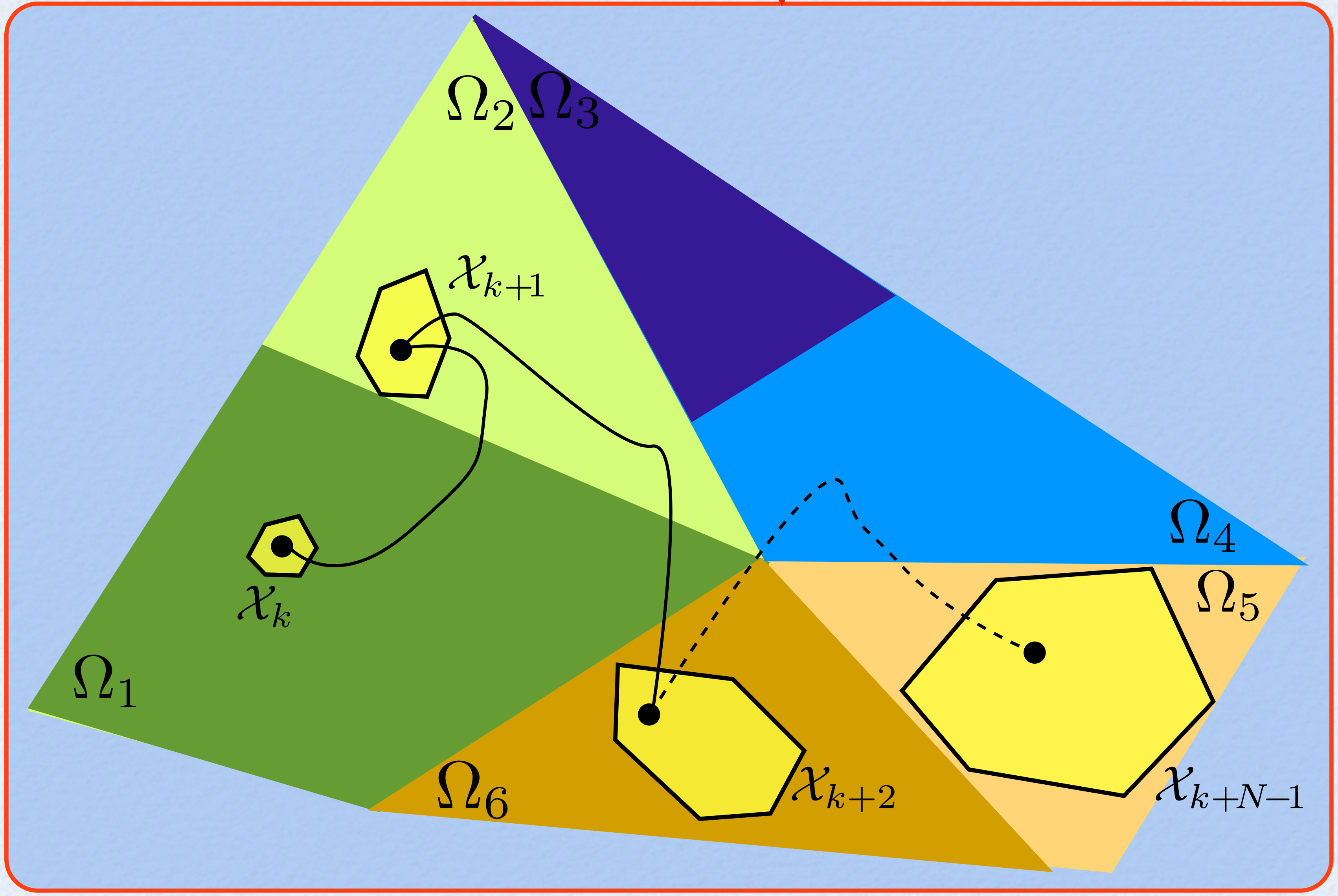


Reconstruct from $k \rightarrow k+T+1$: $\hat{\mathbf{i}}_T$

Optimal control of stochastic hybrid systems

Estimation & Robust Mode Control

- How does Robust Mode Control helps estimation?



Selects for N instants into the future a feasible DMS:

$$\mathbf{i}_N = \{\Omega_1, \Omega_2, \Omega_6, \ldots, \Omega_5\}$$

Optimal control of stochastic hybrid systems

Estimation & Robust Mode Control

- How does Robust Mode Control helps estimation?

Minimize:

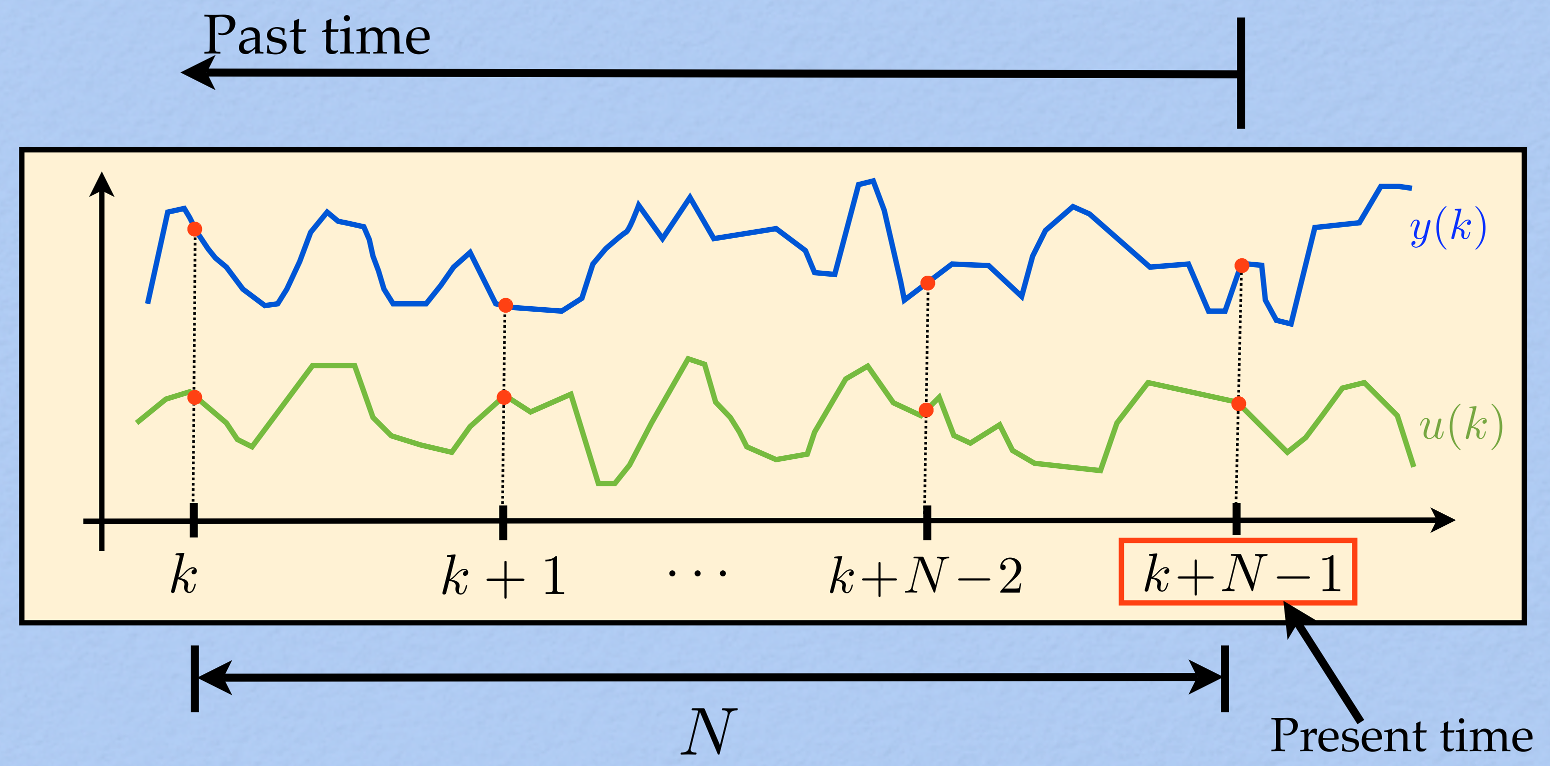
$$\left\|\hat{Y}^{*}_{\mathbf{i}_N}(k|t) - Y_T(k)\right\|^2$$

and obtain:

$$\hat{x}_{\mathbf{i}_N}(k|t)$$

$$\hat{w}_{\mathbf{i}_N}(k|t)$$

$$\hat{v}_{\mathbf{i}_N}(k|t)$$



Use the DMS that resulted from RMC: $\mathbf{i}_N$

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Part IV

EXPERIMENTAL APPLICATION

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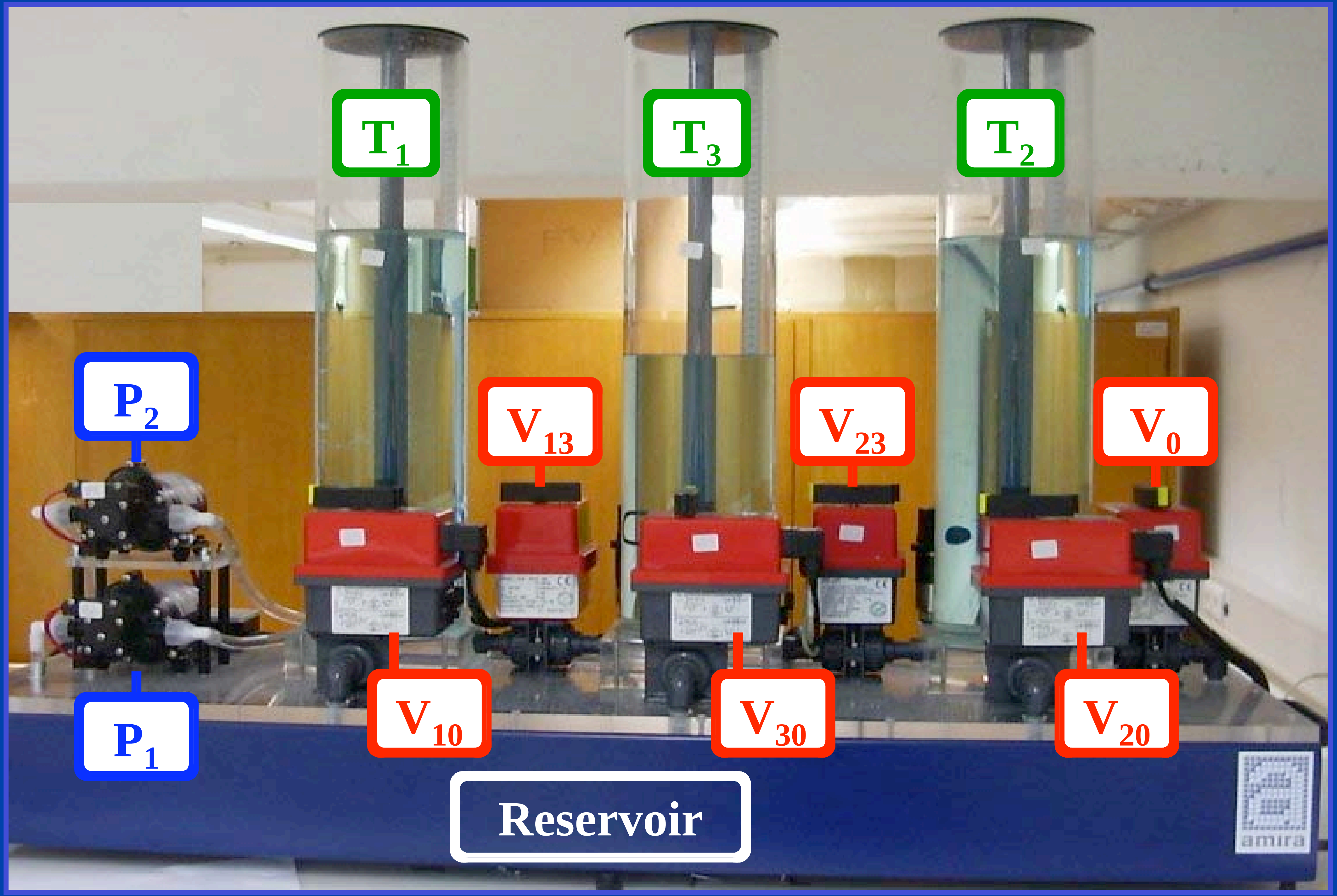
Experimental application

3-Tank experimental setup



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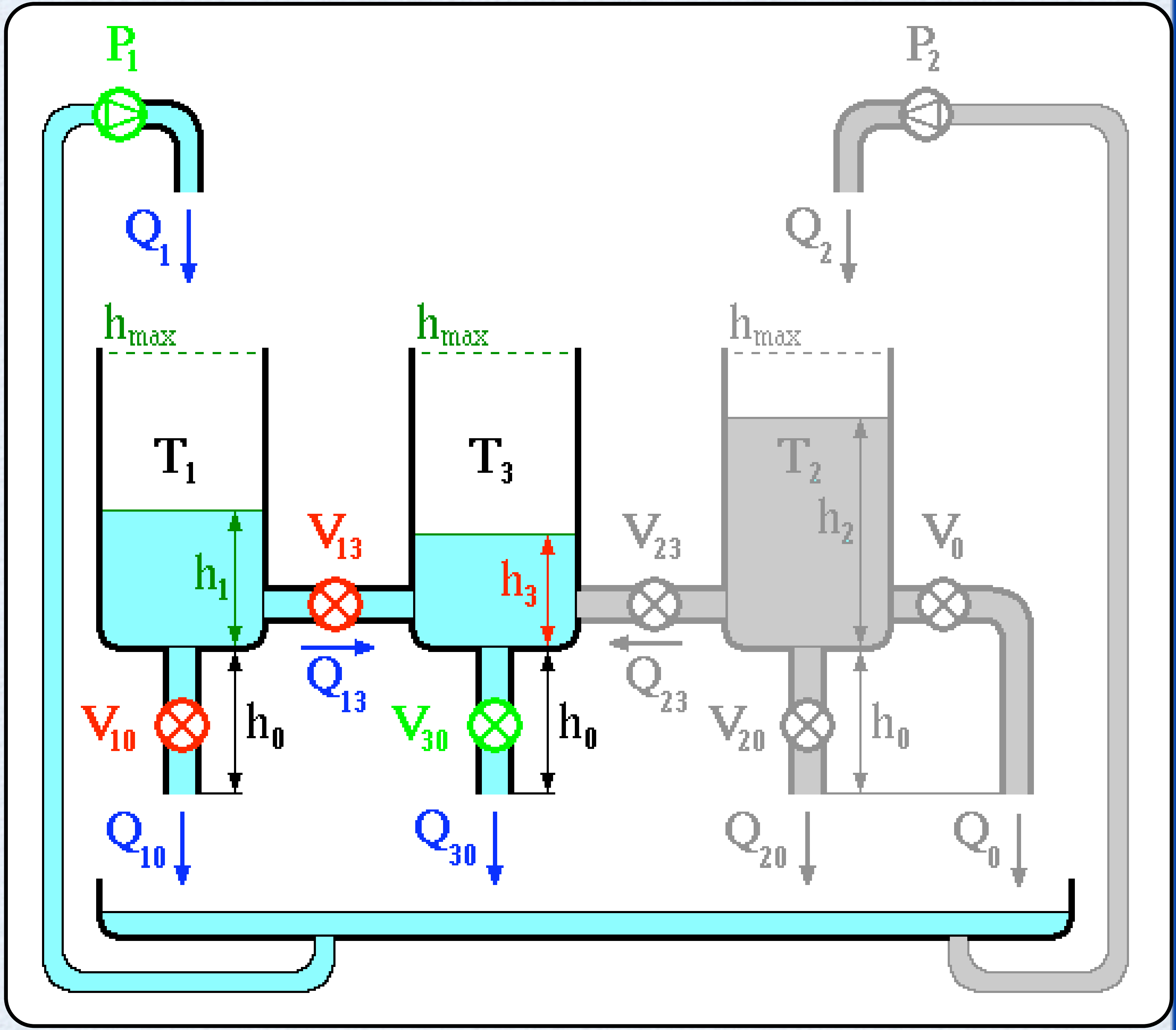
Experimental application

Fault detection on a simplified configuration



- 3 on/off inputs:

  P_1, V_{30}, V_{13}

- 3 discrete faults:

  V_{10}, V_{13}, h_3

- 4 discrete linearization variables.

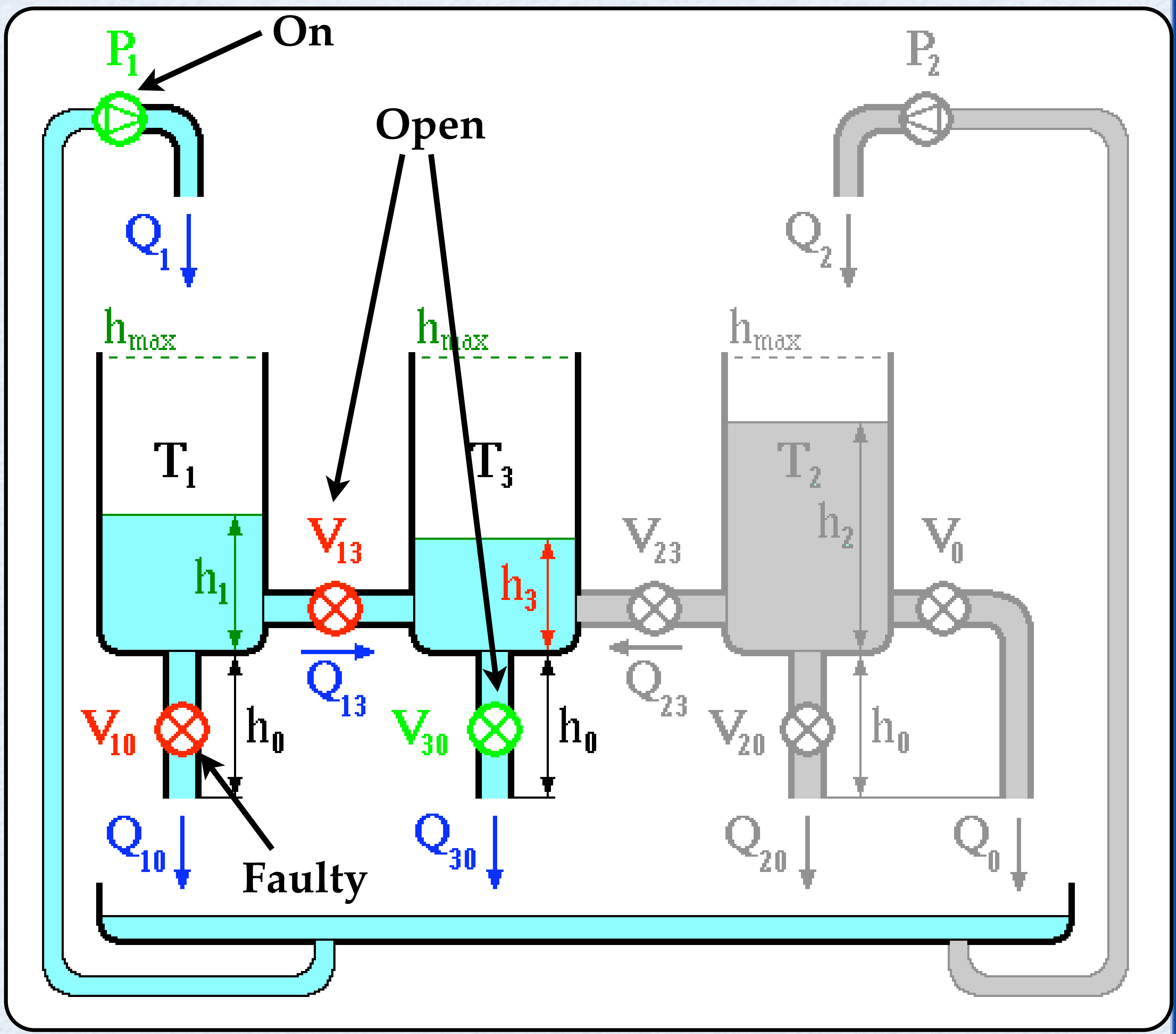
$\Downarrow$

Total number of discrete modes:

$$2^{10} = 1024$$

Experimental application

Fault detection on a simplified configuration



- **Objective:**

  Estimate the discrete mode that indicates a fault on valve:

$$V_{10} = \left\{ \underbrace{\text{closed}}_{\textbf{OK}}, \text{interm}, \text{open} \right\}$$

  based on the output measurements:

$$h_1, h_3$$

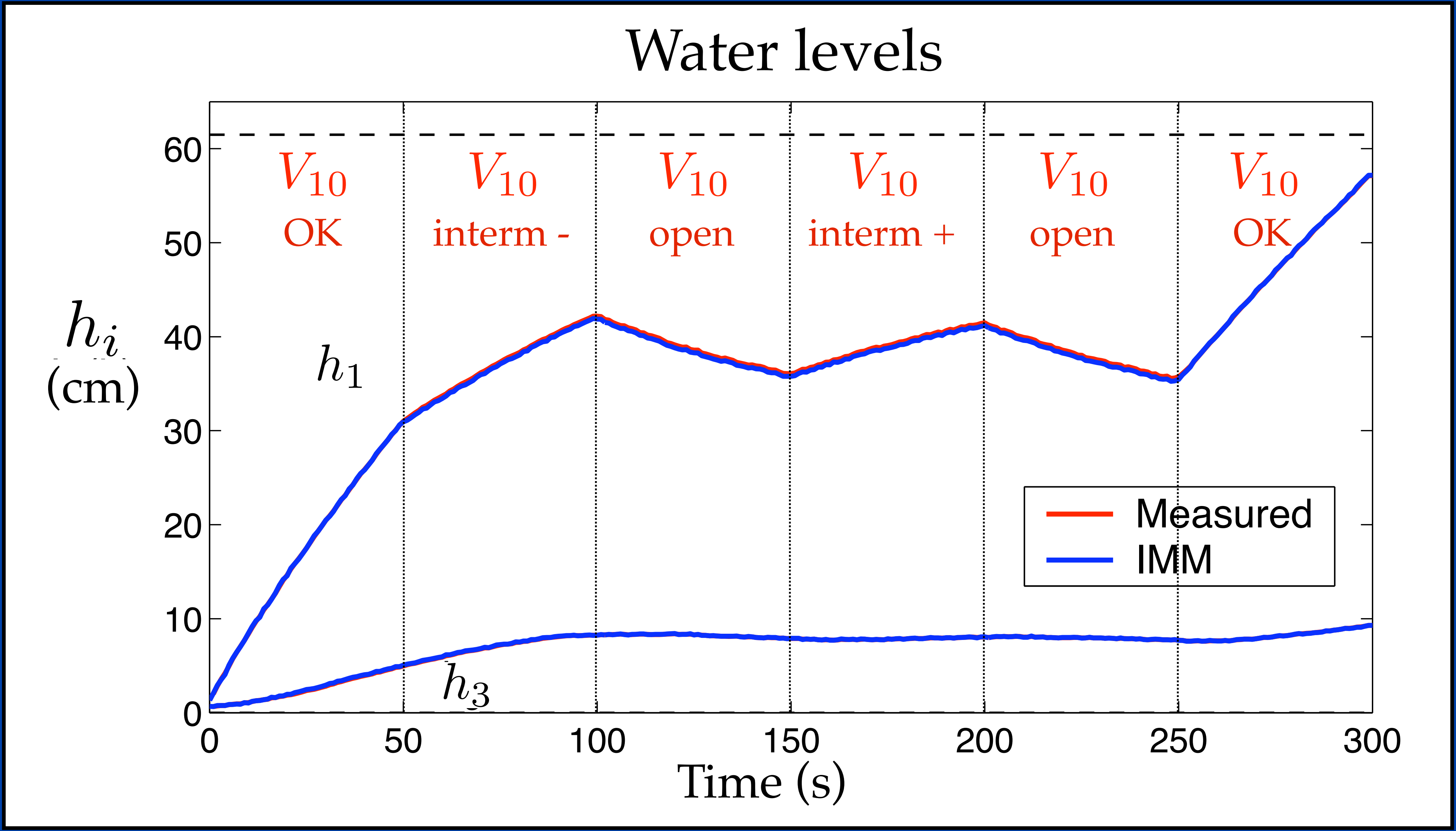
- **Consider** $T = 2$**:**

  $\Longrightarrow$ 1.048.576 DMS
  (feasible = 214.909)

Experimental application

Fault detection on a simplified configuration

- **Continuous state estimation:**



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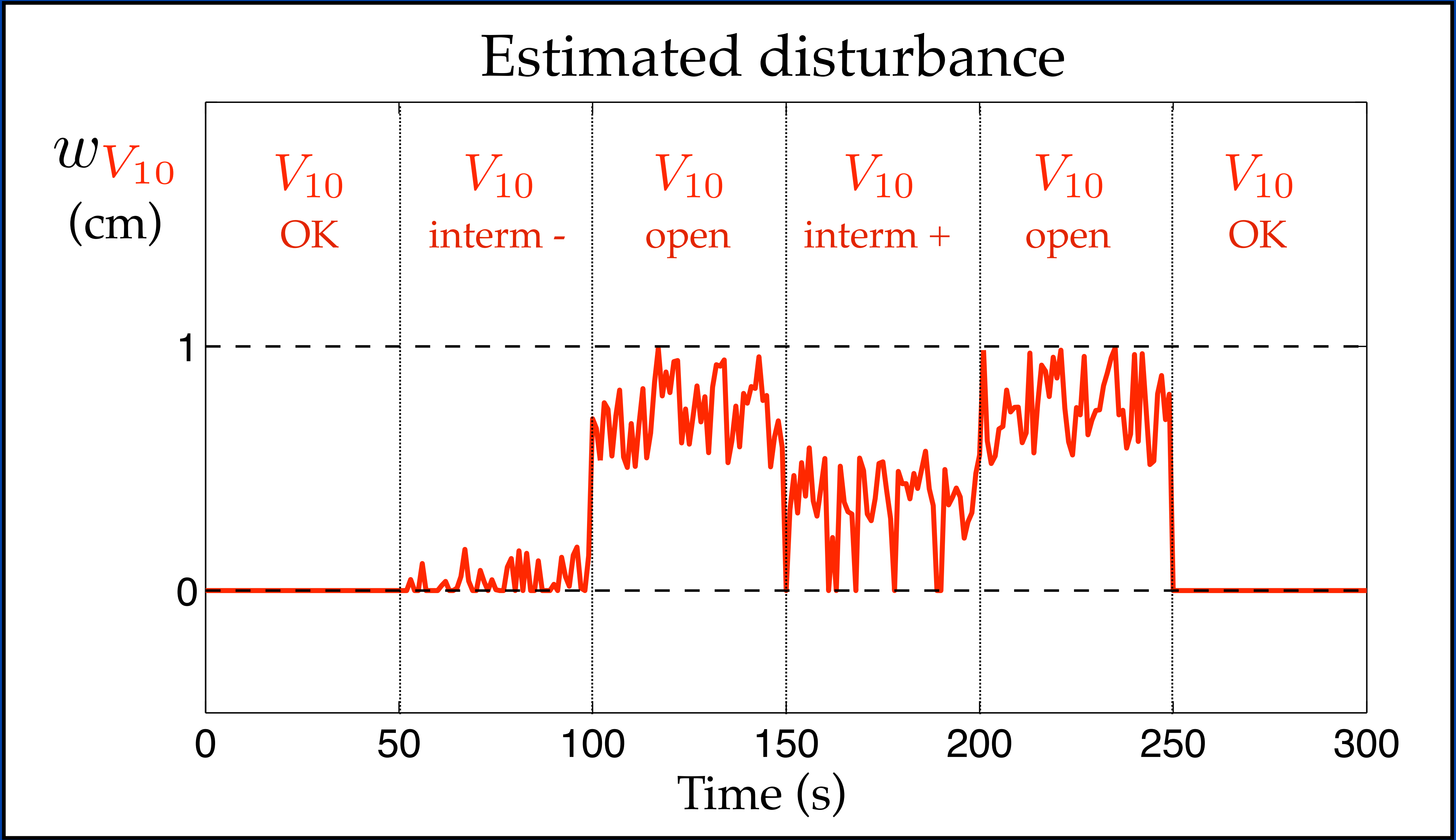
Experimental application

Fault detection on a simplified configuration

- **Continuous state disturbance estimation:**



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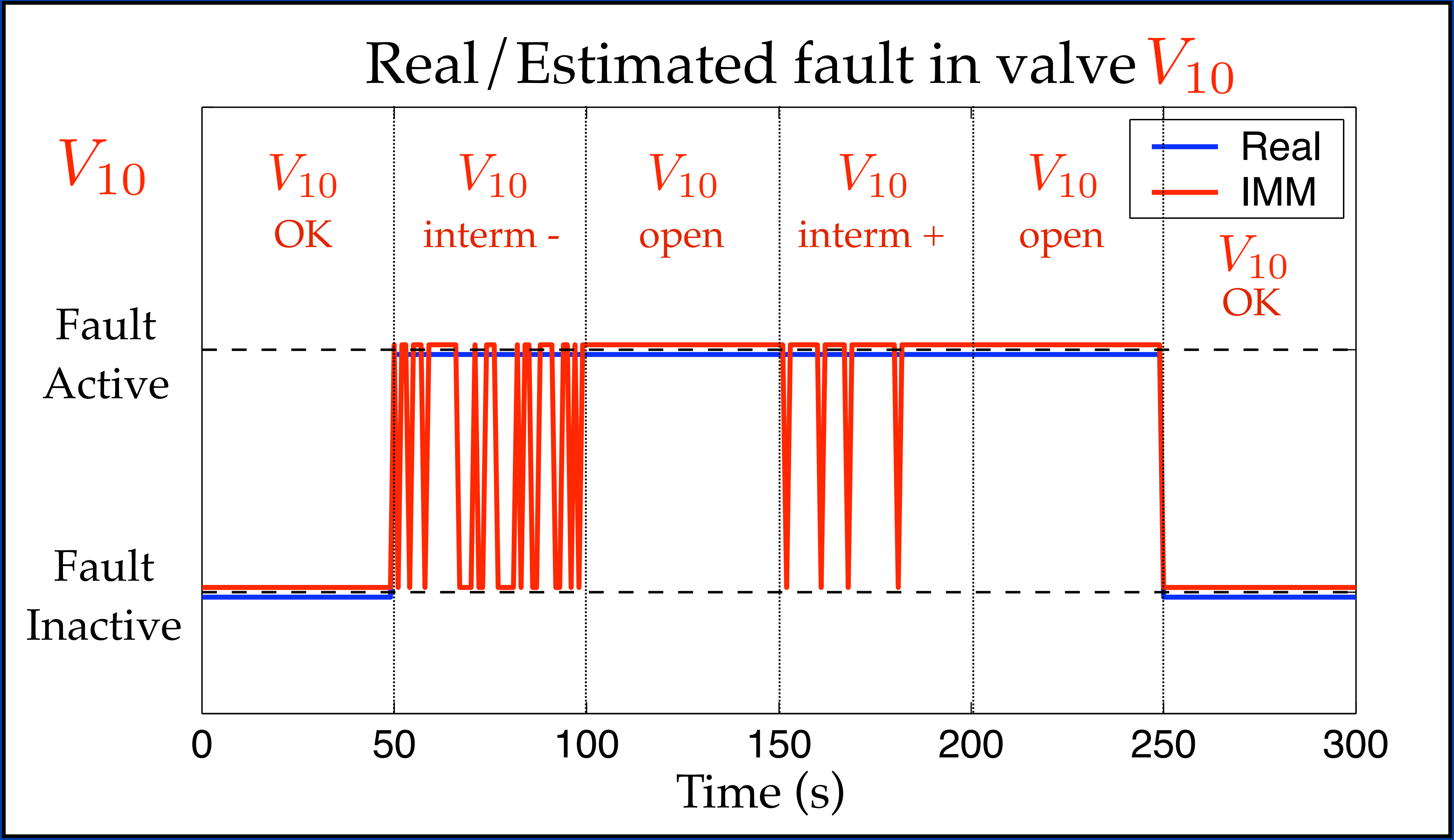
Experimental application

Fault detection on a simplified configuration

- **Discrete mode estimation (fault estimation):**



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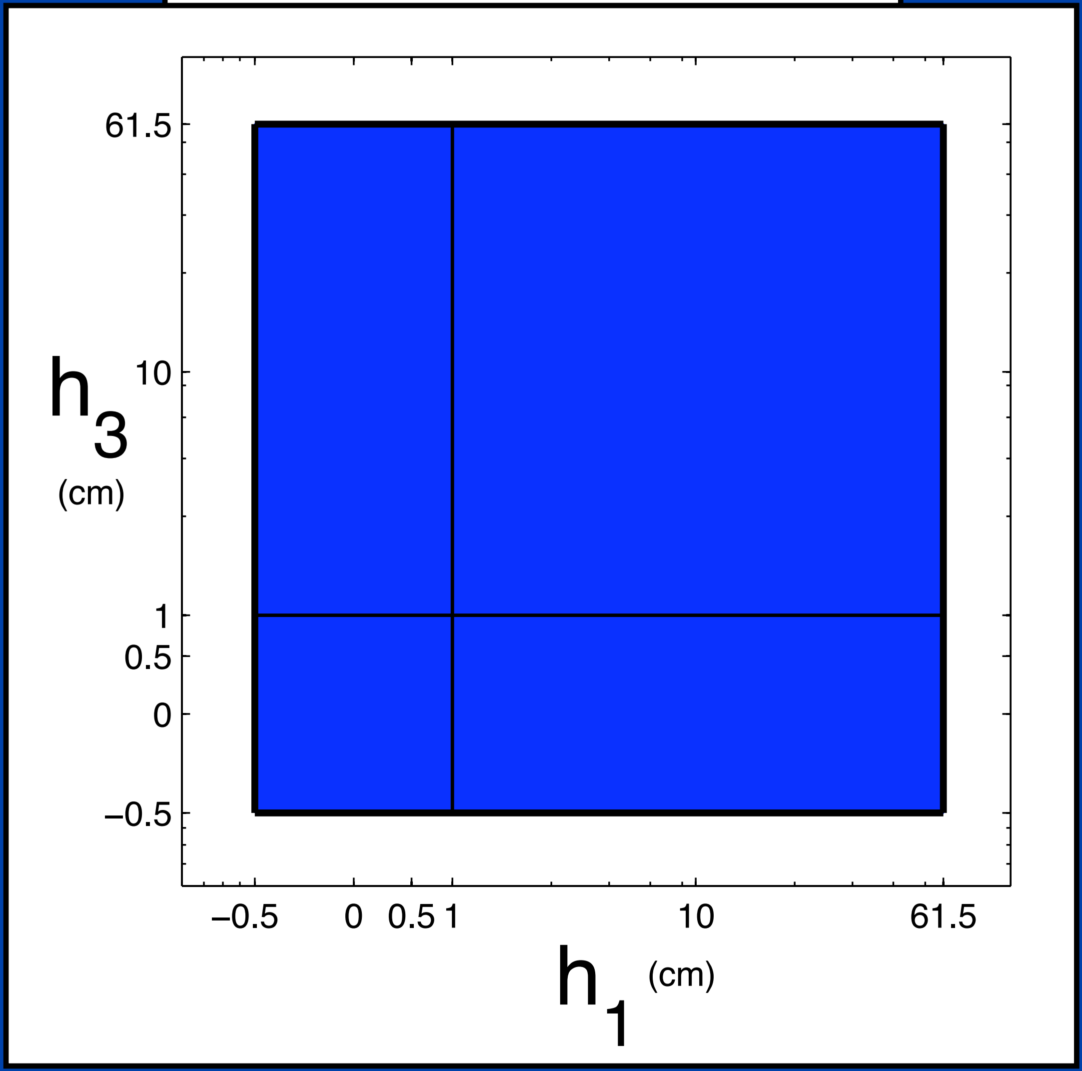
Experimental application

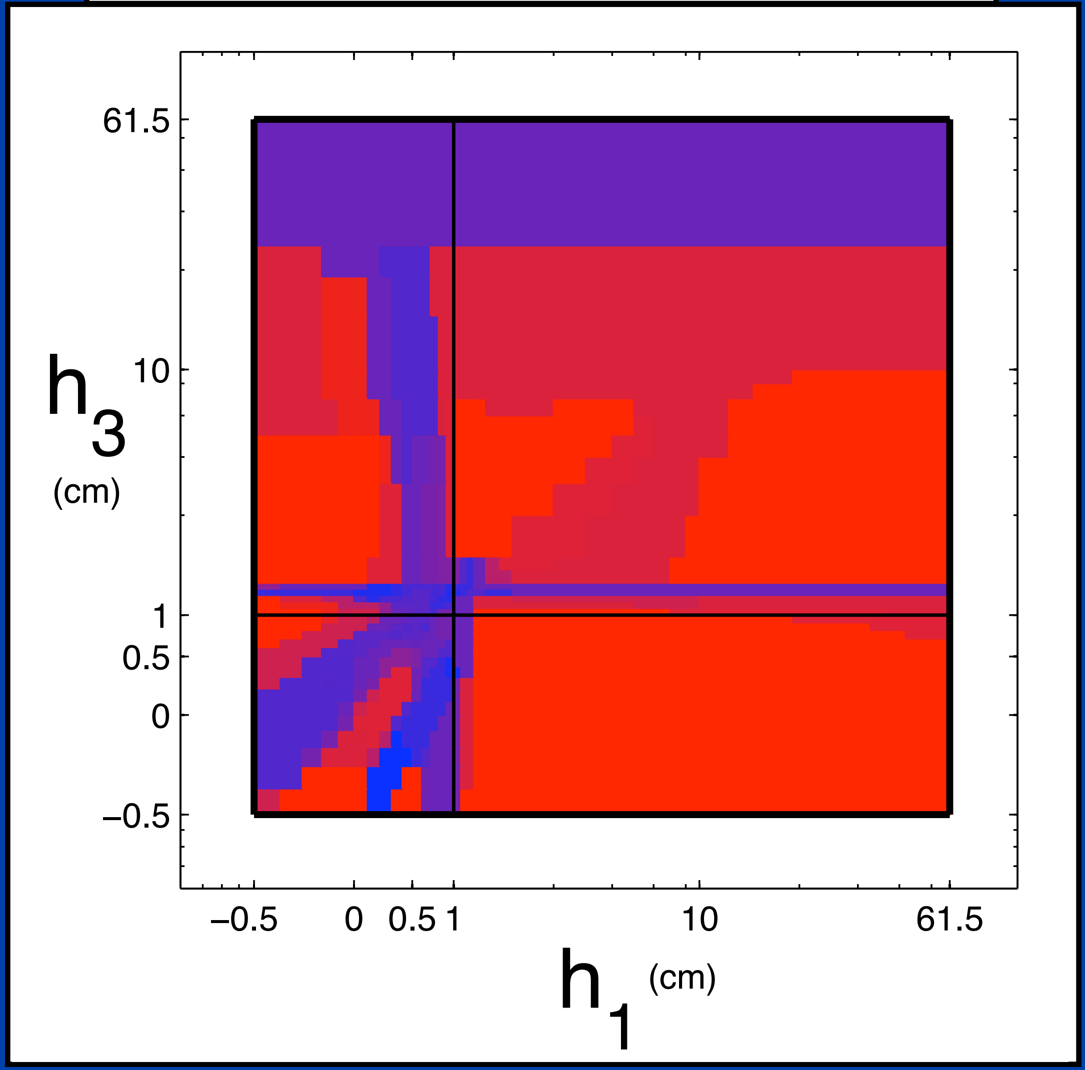
Fault detection on a simplified configuration

- **Probability of discrete mode estimation (fault estimation):**

V_{10} OK or V_{10} open

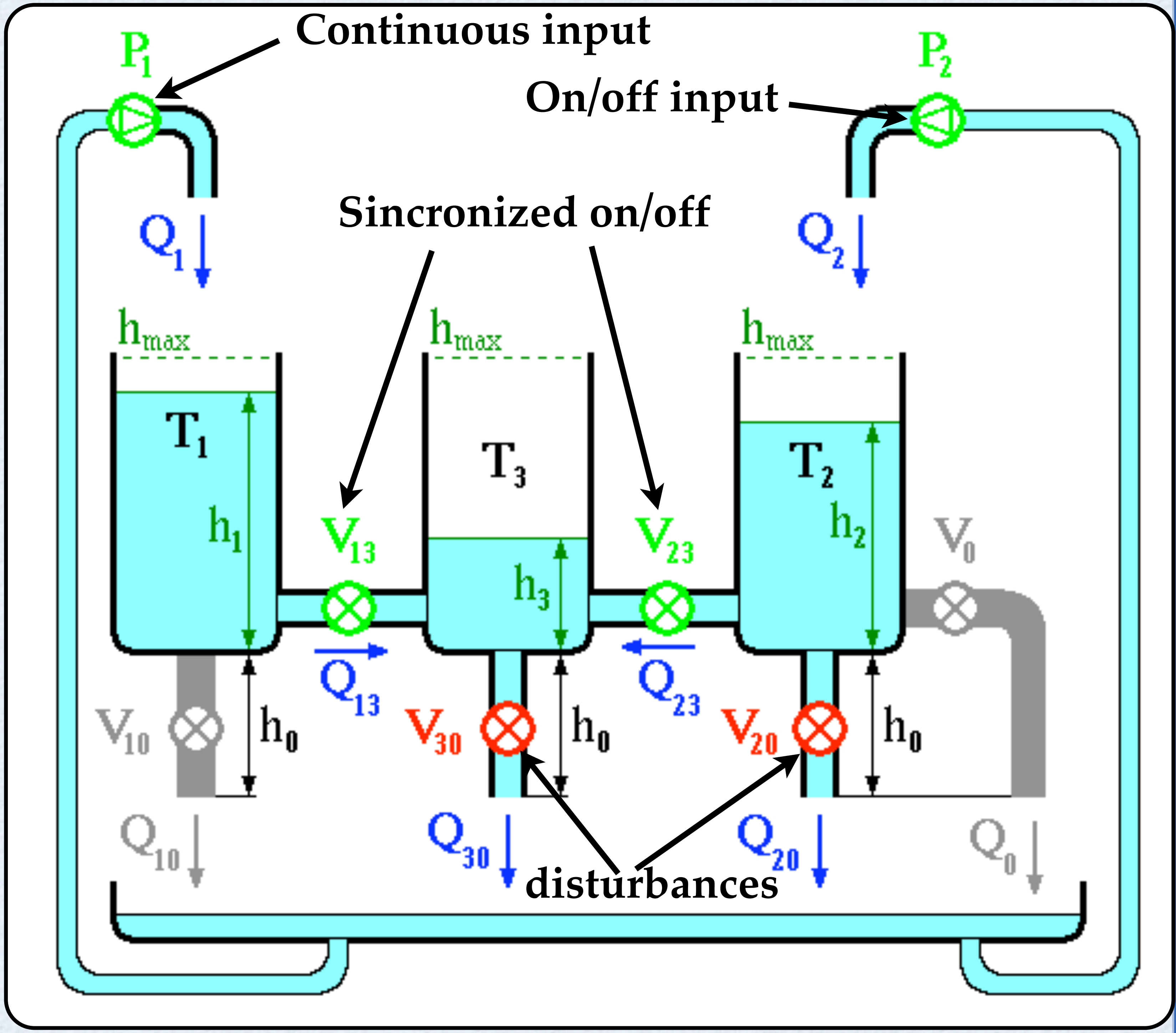


V_{10} interm - or V_{10} interm +



Experimental application

Optimal control with state estimation



- 2 on/off inputs:

  $P_2\ ,u_{\text{mix}} = (V_{13}, V_{23})$

- 50 discrete linearization modes.

$\Downarrow$

Total number of discrete modes:

200

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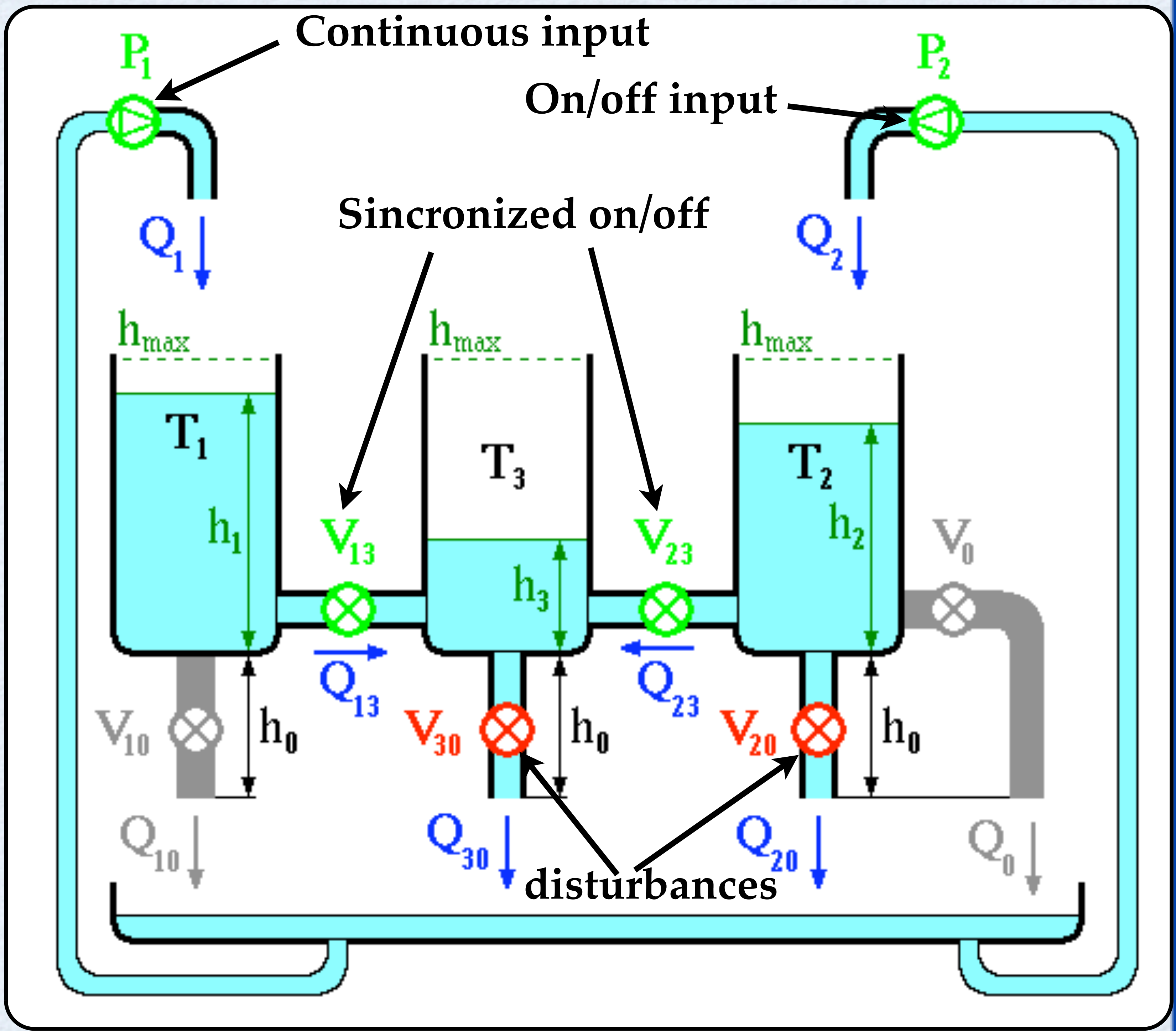
Experimental application

Optimal control with state estimation



- **Objective:**

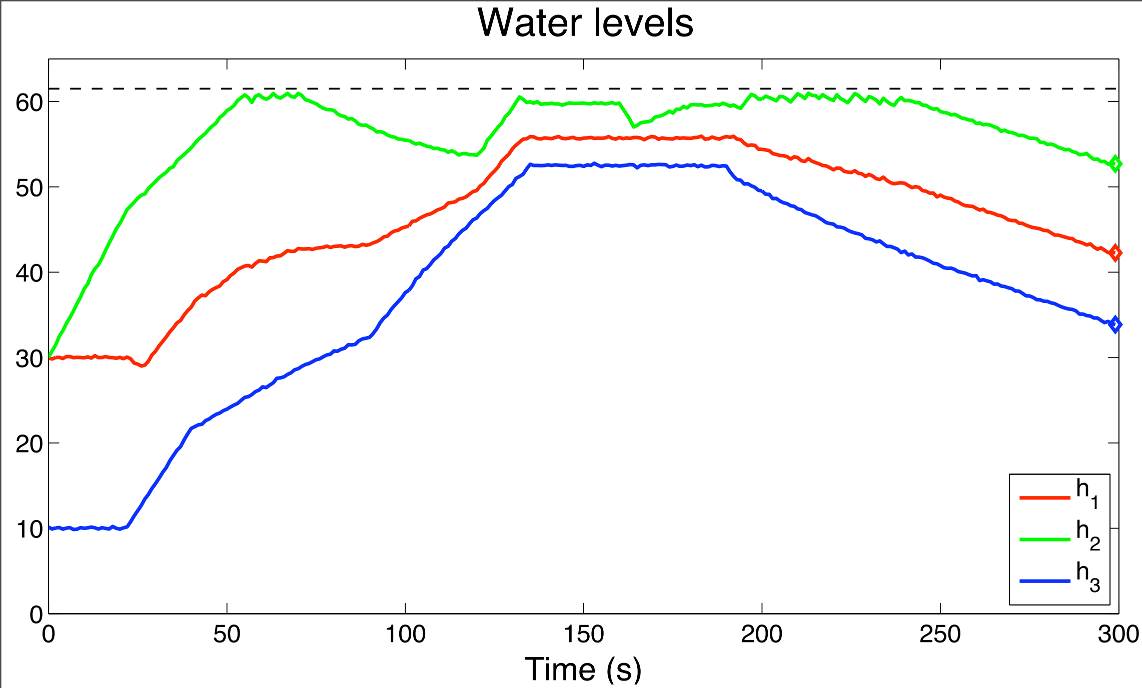
  Maximize h_3 such that:

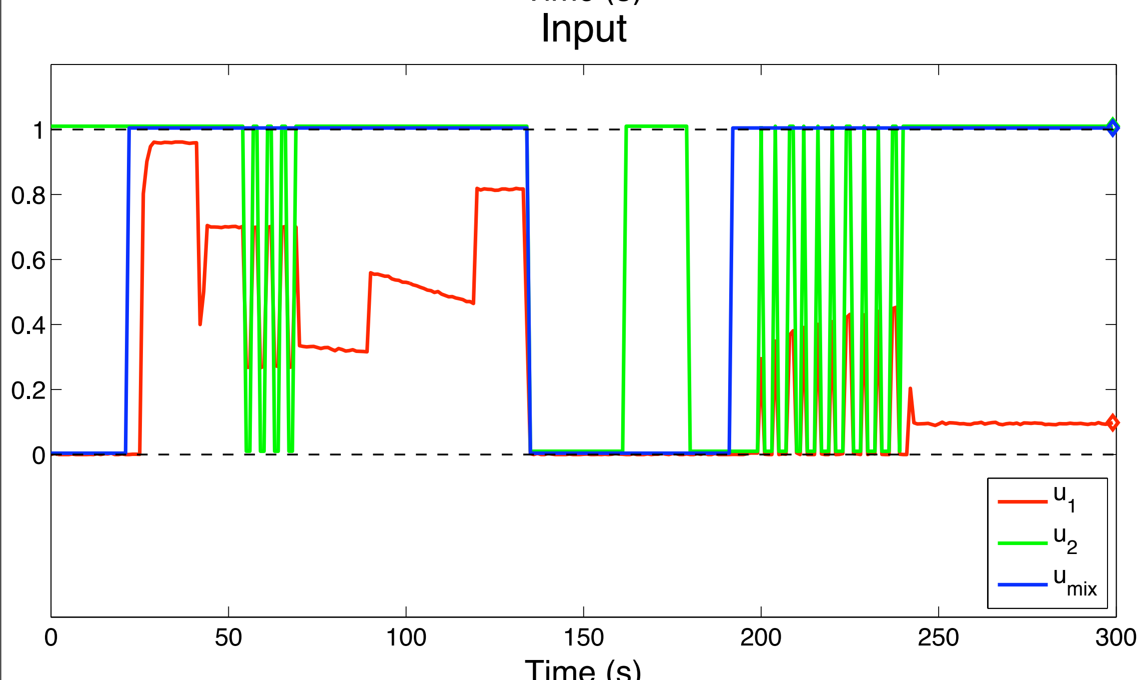
  1) T_1 and T_2 do not overflow.

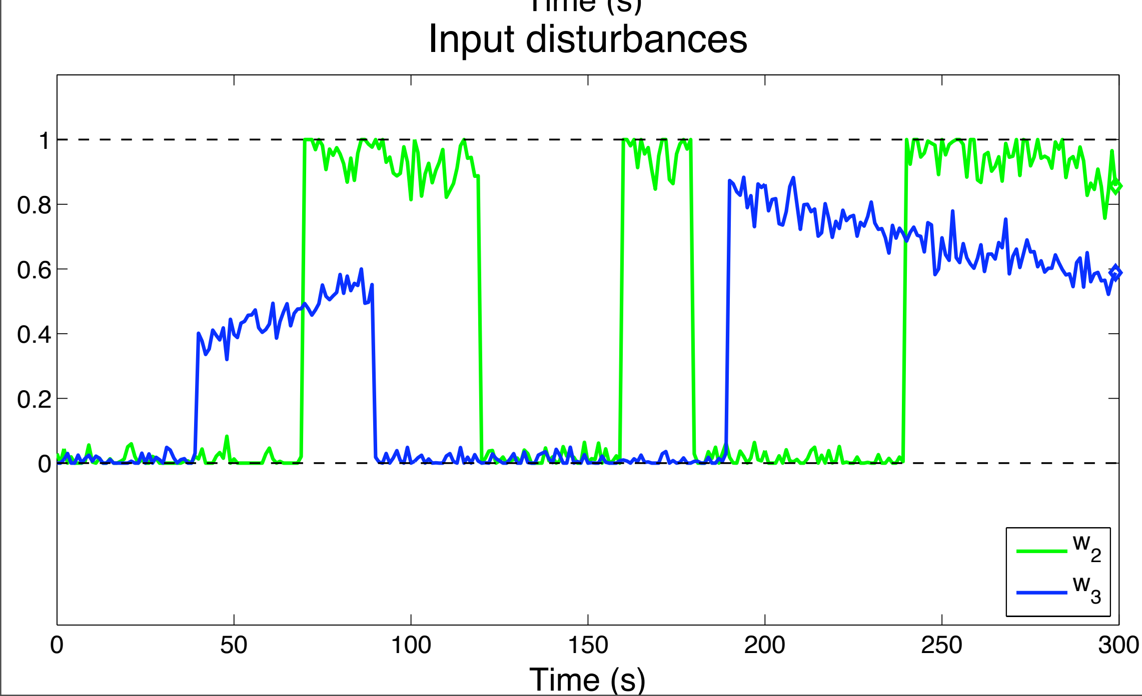
  2) $2 \times Q_{13} = 3 \times Q_{23}$

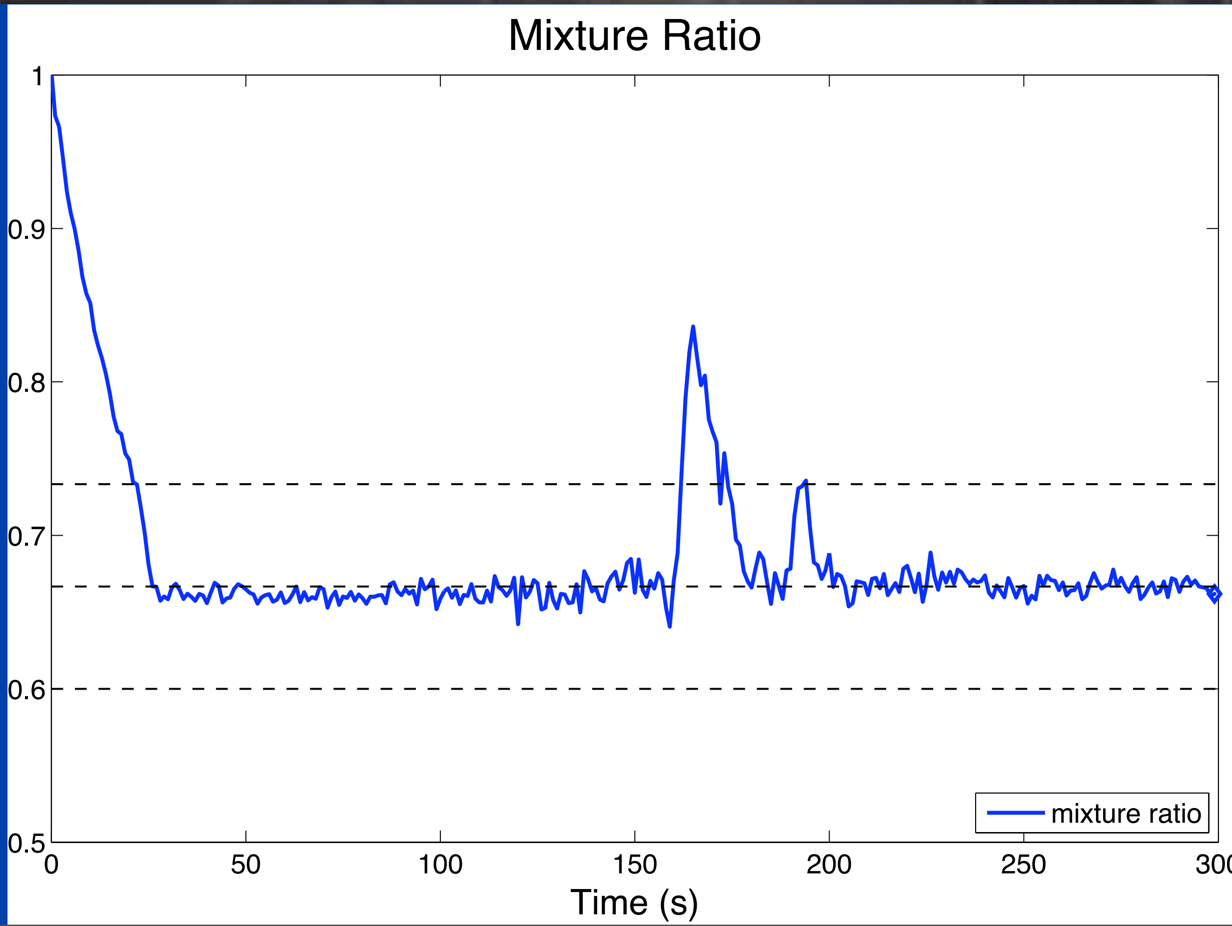
  subject to disturbances:

  V_{30}, V_{20}


Water levels
Time (s)
h1
h2
h3


Input
Time (s)
u1
u2
umix


Input disturbances
Time (s)
w2
w3


Mixture Ratio
Time (s)
mixture ratio

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Part IV

CONCLUSIONS AND FUTURE DEVELOPMENTS

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Conclusions and future developments

- Hybrid systems: powerful tool for the design of embedded systems.
- Hybrid systems: quite complex models, easily explode in the number of variables.
- Simultaneous state and mode estimation: Np complete MIP problem, which size grows exponentially with the number of discrete modes.
- Multi-agent architectures: eliminate redundancy in the model.
- Robust hybrid stochastic control: consider mode uncertainty with known state, instead of Robust Mode Control. Merge of both?
- Applications: still finding important applications, like control over networks, platoon control, humanoid robotic applications, cooperative agent control, biological systems.